MATHEMATICS OF ACCOUNTING

BY

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Preface

It is now somewhat more than twenty years since the publication of the first edition of *Mathematics of Accounting*. Those familiar with the original edition, and the revised edition ten years later, will find that in the present revision the sequence of subjects has been considerably changed; the treatment of some subjects has been amplified; certain other subjects have been added. Some problems have been changed; new problems and review problems have been included.

Subjects new to Part 1 are: Factors and Multiples, Business Insurance, and Payroll Records and Procedure. The chapter on Graphs has been expanded to include Index Numbers.

Part 2 has been enlarged to include chapters on the following: Permutations and Combinations, Probability, Probability and Mortality, Life Annuities, Net Premiums, and Valuation of Life Insurance Policies.

Algebraic formulas have been changed to conform to standard usage, while arithmetical substitutions and detailed solutions have been retained. Those desiring to work entirely from the tables in the appendix will find table references in the detailed solutions.

Grateful recognition is here given to those who have taught from the previous editions and who have responded with suggestions for this revision.

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Part I

PART I

CHAPTER 1

Fundamental Processes and Short Methods for the Accountant

Addition. Addition is the process of combining numbers of the same denomination. Quantities of such unlike measures as *dollars* and *yards* cannot be added; but quantities like *yards, feet,* and *inches* can be changed to like numbers and then added. *Like* numbers are numbers that express the same kind of units. The *sum* is the number resulting from adding two or more like numbers, and the *addends* are the different numbers to be added.

Addition is the most fundamental of all numerical operations. It is essential that the clerk, the businessman, and the accountant be able to add with precision and rapidity. The ability to recognize the sums of numbers instantly is acquired by constant practice and careful study.

Drill tables. Practice adding the columns of numbers in the following table until you can complete the operation in twenty-five seconds, without error. State sums only; that is, do not repeat the numbers to be added.

5	8	1	6	5	4	5	9	6	7	2	9	4	8	9
1	2	1	3	3	2	2	7	4	7	1	6	4	6	2

7	9	6	3	7	5	9	4	3	9	5	9	8	4	8
4	9	5	1	2	5	4	3	2	8	4	5	8	1	7

7	8	6	6	8	9	2	7	8	8	7	6	3	7	9
3	4	2	1	5	3	2	5	1	3	1	6	3	6	1

Practice stating the sums of the following columns of numbers until you can do all of them correctly in less than two and a half minutes.

3	4	2	3	4	3	3	5	5	7	6	5	2	3	5	3	8
2	2	2	3	3	3	2	3	3	2	3	3	2	2	2	3	2
4	5	8	6	4	4	5	7	6	9	8	5	4	8	5	7	8

[*Drill table continued on next page.*]

5	6	7	2	3	2	7	2	8	4	7	4	4	6	3	3	2
4	3	1	2	2	2	2	2	2	3	2	4	2	2	2	2	2
6	6	8	9	7	6	8	7	9	6	7	5	4	6	6	3	5

4	5	8	4	9	6	3	5	3	4	5	4	8	4	5	6	8
4	4	3	2	1	2	3	2	2	3	2	2	1	2	3	2	3
7	5	9	8	9	7	8	6	9	9	7	7	8	9	8	9	8

6	7	4	5	7	5	7	6	8	8	6	3	4	6	3	9	6
5	4	2	2	6	4	1	4	4	1	3	3	3	1	3	2	1
6	7	6	9	8	8	7	9	8	9	9	5	7	9	9	9	8

9	5	8	8	6	4	8	7	8	6	5	5	7	7	7	7	6
8	2	8	6	3	3	5	6	7	5	5	3	7	5	4	6	4
9	8	9	9	7	5	9	9	8	7	8	9	9	9	8	7	6

7	6	7	4	7	5	8	9	6	4	5	6	9	5	9	6	6
4	4	3	1	4	6	4	4	3	4	6	7	5	5	5	5	1
8	7	8	9	9	9	8	9	8	8	7	7	9	9	9	8	9

7	8	8	7	9	6	5	7	6	8	9	7	2	7	4	4	5
3	4	5	5	3	6	5	4	6	7	6	5	2	3	4	4	5
9	9	8	7	9	9	6	9	8	9	9	8	3	7	8	6	7

Streamline addition. Omit unnecessary words: that is, do not name the number to be added; name only the sum.

5
7
8
4
3
2
9
38

In the example at the left, a common way of adding would be (commencing at the top): 5 and 7 are 12, 12 and 8 are 20, 20 and 4 are 24, and so forth. Instead of adding in this manner, proceed to the answer by saying (mentally), "12, 20, 24, 27, 29, 38."

Drill table.

3	4	2	6	7	5	1	8	3	5
2	5	9	8	9	3	4	7	2	8
7	3	7	7	2	6	8	6	7	1
4	6	6	9	5	8	6	9	6	3
5	9	3	3	8	7	2	4	9	9
6	8	4	2	6	2	5	7	4	6
1	5	8	4	3	9	7	5	3	7

Combinations whose sum is 10. Combinations of two or more numbers whose sum is 10 are of frequent occurrence. When these combinations are recognized, addition may be shortened by adding such combinations as 10.

4
7
3
2
5
9
5
3
2
40

In this example, the addition may be performed as follows: (commencing at the top) 14, 16, 21, 30, 40.

Or, it may be added in this manner: 11, 21, 30, 40.

Do not try to form combinations. Unless they are instantly recognized, add the numbers in the regular manner.

Drill table.

7	3	5	3	2	6	4	1	9	2
2	5	6	8	8	9	2	5	8	7
1	6	2	2	5	5	4	9	7	4
5	4	7	7	4	4	7	8	3	5
6	9	1	6	3	1	5	3	6	1
4	8	4	3	7	7	3	7	3	7
9	2	8	1	5	8	2	6	1	5
8	7	2	8	2	2	6	4	2	5
2	4	5	7	3	9	8	2	5	9

Adding where the same number is repeated many times. In obtaining averages, in adding statistics, and in other work involving addition, often the same number is repeated many times. Use multiplication to save time in adding.

724
785
773
748
696
687
679
5092

In this example, 7 occurs four times and 6 occurs three times in the third column. The sum of the third column may be found as follows:

Carried	4
4×7	28
3×6	18
	50

The work is actually performed mentally thus: 4 (28) 32, (18) 50. Where the columns are long, a side calculation may be necessary.

Drill table.

68	284	34.86	23.56	48.34	71.53
63	273	33.75	23.95	47.56	72.37
64	281	32 86	24.72	39 85	72 48
67	311	31.29	25 31	38.64	69 95
59	314	34 36	26 54	45 58	68 83
54	321	32.75	31 72	39 95	68 44
57	318	33.95	32.69	42.74	67.93
56	319	36.87	33 47	38.56	71.59

Group addition. The most practical method of adding is to group or combine two or more figures mentally, and to name results only.

	Mental Steps			
5				
4	9			
7				
3	10	19		
8				
6	14		33	
1				
2	3			36
36				

Instead of saying, "5 and 4 are 9, and 7 are 16, and 3 are 19, and 8 are 27, and 6 are 33, and 1 are 34, and 2 are 36," simply *think*, "9, 19, 33, 36."

Drill table.

6	4	8	5	3	7	9	3	8	6	4	5	7	3	8	7	4	2	9
3	6	5	2	4	8	7	6	3	2	5	7	6	3	9	3	8	4	7
7	3	6	2	9	8	4	6	7	4	5	7	6	2	4	6	9	7	4
7	6	3	5	8	7	5	3	2	5	9	7	3	8	4	6	8	2	7
8	2	4	8	3	7	4	6	5	4	9	7	6	3	5	8	7	3	6
3	8	6	2	4	7	8	6	3	5	8	6	3	4	7	6	2	4	8
5	7	3	7	3	5	8	2	3	7	6	3	4	8	6	2	3	5	9
4	3	7	4	2	7	8	6	4	7	6	4	8	3	9	5	8	2	1
5	5	9	9	8	6	3	6	7	7	7	7	5	3	4	2	1	7	3

Addition of two columns at a time. Two columns of figures may be added at the same time, as shown in the following illustration:

	Mental Steps			
56	*Tens*		*Units*	
28	(*1*)	76	(*2*)	84
43	(*3*)	124	(*4*)	127
21	(*5*)	147	(*6*)	148
148				

	Explanation		
	Tens		*Units*
(*1*)	56 and 20 = 76	(*2*)	76 and 8 = 84
(*3*)	84 and 40 = 124	(*4*)	124 and 3 = 127
(*5*)	127 and 20 = 147	(*6*)	147 and 1 = 148

Drill table.

79	82	24	37	65	39	28	28
48	84	33	44	81	58	39	59
81	95	46	53	42	48	23	86
15	83	52	66	73	73	37	63

Recording addition by columns. Accountants are subject to interruptions, but the time required to re-add a column of figures for the purpose of picking up the carrying figure may be saved if the total of each column is recorded separately. The separate column totals are also convenient to use in checking the work; for instance, if in a final summary of additions there is an error of $100.00, the hundreds' columns of the subtotals may be verified quickly without the necessity of re-adding all the columns.

Example—Method 1	*Example—Method 2*	*Example—Method 3*
4572	4572	4572
3986	3986	3986
2173	2173	2173
5911	5911	5911
2765	2765	2765
4937	4937	4937
24	**24**	24
32	**34**	34
40	**43**	43
20	**24**	24
24344		24344

Explanation 1. Add each column separately, setting the sums one place to the left, as in the example. After the last column has been added, add the individual sums in regular order; that is, from right to left.

Explanation 2. In Method 2, a little time is saved by adding to each column the number carried from the column at the right.

Explanation 3. Method 3 differs from Method 2 in the writing of the columns' sums. It is somewhat easier to write the sums one below the other. This cannot be done in Method 1 because carrying figures are not used, and another step is required to complete the answer: that is, finding the grand total of the units, tens, hundreds, and so forth.

A modification of the third method is useful in adding columns of dollars and cents.

$	644	22
	821	94
	314	26
	712	84
	976	54
	592	28
$4,	062	08

The total, $4,062.08, is obtained by adding each column separately as explained under Method 3. The computation will appear as follows, the purpose of the horizontal lines being to separate cents from dollars, and hundreds from thousands.

Sum of the first column..........................	28
Sum of the second column, 28 plus 2, carrying number ...	30
Sum of the third column, 19 plus 3, carrying number	22
Sum of the fourth column, 24 plus 2, carrying number ...	26
Sum of the fifth column, 38 plus 2, carrying number	40
As there are no more columns, write the carrying number.....	4

The total, $4,062.08, is obtained by reading the numbers at the right, commencing at the bottom.

Problems

1.	2.	3.	4.	5.	6.	7.	8.
5273	5126	7952	1395	3688	$367 98	$786 42	$498.57
2191	8497	2975	2764	4932	421.74	518 49	822 56
8437	7934	8675	8351	7563	281.34	946.72	753 86
3426	9783	8437	6248	2898	633 46	881.92	629.75
7139	9126	2975	5347	6598	855.91	542 37	367.43
7895	8751	3826	4586	8877	769 25	787.66	521.54

Practice Problems

Average weekly earnings from payroll reports.

1.	2.	3.	4.	5.	6.	7.	8.
33 20	35 72	30 85	33.20	28 28	37 61	22.53	23.25
25.13	29 88	21.99	35 72	28 24	29.97	35 62	35 25
37.41	39 24	35 31	42 28	22 61	31.36	16.22	32 18
31 65	33 47	28 89	31.56	21 46	21.34	31.91	37 17
31.40	35.29	27.89	29.82	23.91	33.34	14.16	35 99
22 93	23.22	19.11	37.33	22 41	31.78	27.14	36 66
32.16	33.49	19.15	28 97	17 99	34 73	31 62	35 96
26 37	28 34	18.96	54.61	25 21	26 74	33 84	37 37
36 52	33.64	30.73	39 06	33.26	22 88	30 89	18.17
32 05	34 60	30.70	28.29	28 72	28 09	39 04	29 64
32 58	26 49	18.95	26 87	19 49	28 20	15 76	15 82
23 60	28.81	33.45	27.01	18.64	20 80	20 87	29 87
23 44	37.92	31 60	39.52	37.12	37 53	38 72	27 84
36 37	41.54	30 41	41.86	31.04	28 94	37.16	36 83

Tabulation of advertising lineage.

9.	10.	11.	12.	13.	14.	15.	16.
26,228	29,207	22,107	14,849	10,049	57,104	13,022	10,755
13,818	17,588	15,977	11,966	14,745	71,075	15,223	16,850
27,122	28,267	39,082	36,021	8,562	119,035	17,058	15,573
17,077	15,095	9,644	7,888	5,575	28,857	18,048	10,259
32,094	36,072	23,449	19,634	12,376	39,190	26,174	19,635
32,936	32,835	18,930	15,033	2,175	16,085	28,169	24,572
21,499	18,116	46,520	43,778	7,531	15,484	14,949	15,057
20,655	24,094	25,140	19,271	8,650	28,192	24,478	19,445
22,338	10,365	8,015	13,412	15,530	14,711	22,175	24,493
13,412	60,475	38,795	93,323	23,680	22,865	23,680	37,335

Addition of dollars and cents, irregular items.

17.	18.	19.	20.	21.	22.
139.10	86.35	209.80	45.40	86.35	1,955.05
12.65	52.67	44.82	34 20	52 67	531.03
10 57	44 00	37.45	28 66	42 57	442.85
50.05	208.33	127.29	135 68	208 33	2,148 74
1,275.48	394.68	4,151.36	945 21	878 52	30,149 39
260.73	64.72	24.94	2.72	111 56	112 64
7.81	6.29	.72	118.61	6 46	509.74
78.13	27.33	71.97	.75	44.77	27.02
2.50	.62	12.09	32.49	.69	153 07
111 82	27.65	.35	8.58	3 07	1,512.34
29.53	7.33	160.31	33 55	70 63	1,002.90
54.53	16.29	45 15	8 45	5 37	282 51
15.36	4.59	128.60	24.85	35.15	66 66
147 62	111 52	41.51	138.34	9.98	146.43
5 27	59.68	46.43	92.54	214.34	641.51

Practical applications. In the following problems will be found examples of business records requiring addition for the completion of the record.

Problem 1

In this problem, cash register tapes provided the source of the entries on Form 1. As the sales were registered, the classification was imprinted on the tape. At the end of the day, the classified items appearing on the tape were accumulated on Form 1, and the totals transferred to Form 2. At the end of the week, Form 2 was added; at the end of the month, the weekly totals were accumulated to monthly totals. Thus, sales for the month were analyzed by departments or classes.

Add the columns on Form 1 (Saturday's sales), transfer the totals to Form 2, and find the total sales for the week.

Form 1

Candy	Cigars	Soda	Drugs	Own Remedies	Patent Medicines	Toilet Articles
.35	.10	.10	.45	75	1.25	.35
1.25	.25	.15	1.64	.45	.50	1.15
.80	.15	.20	.10	.15	1 50	.80
.45	.25	.45	.75	1.25	.89	2.65
.75	.50	.10	1 50	.90	33	.75
90	25	.50	1.75	1.65	2.35	1.85
4.50	1.[illegible]	1.5[illegible]	[illegible]	3.1[illegible]	6.[illegible]	7.55

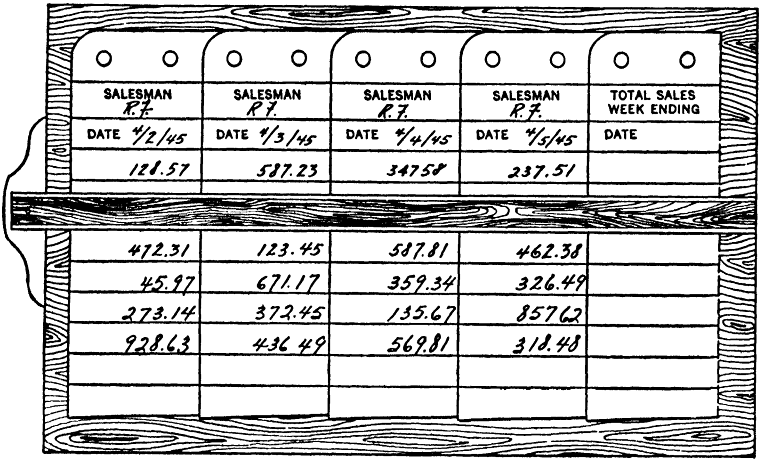
Form 2

Day	Candy	Cigars	Soda	Drugs	Own Remedies	Patent Medicines	Toilet Articles	Totals
Mon.	12 65	19 15	3 95	27.63	4 18	9 85	5.00	82.41
Tues.	8 50	16.10	6 80	33 98	2.47	12.20	3 65	83.70
Wed.	11 25	8 75	4 50	15.20	1 75	2 55	10.45	54.45
Thurs.	9 65	4 25	2.55	7 65	2 85	4 86	4.63	36.44
Fri.	10 35	5.55	3.75	12.84	3.68	5.49	3.85	45.51
Sat.	4.50	1.50	1.50	6.19	5.15	6.52	7.55	33.24
	57.[illegible]0	55.30	23.05	103.49	[illegible]	41.17	35.13	335.79

Form 2 is self-proving—that is, the sum of the daily totals must equal the sum of the departmental or classification totals.

Problem 2

The "peg board" is used for accumulating numbers having to do with many kinds of information. The numbers are entered on narrow forms which are attached to the "peg board." The forms are held in place and cross extension as well as "footings" are thus permitted.



In the following example, this arrangement is used to accumulate total departmental sales made by a salesman.

Salesman R. F.	*Salesman* R. F.	*Salesman* R. F.	*Salesman* R. F.	*Salesman* R. F.	*Total Sales*	*Dept.*
Date 4/2	*Date* 4/3	*Date* 4/4	*Date* 4/5	*Date* 4/6		
128 57	587.23	347 58	237 51	637 82	....	1
645.39	321.69	123 63	563 85	495 71	....	2
362 45	847.86	219 23	149 27	826 45	....	3
472 31	123.45	547 81	462 38	718 26	....	4
45 97	671.17	359 34	326 49	534 58	....	5
273.14	372 45	135 67	857 62	149 17	....	6
928.63	436 49	569 81	318 48	529 32		7
2856.46	3360.34	2303.07	2915.60	3891.31	....	

(**a**) Find the total of each day's sales.

(**b**) Find the total sales for each department.

The answer in the lower right corner proves the work.

Subtraction. Subtraction is the process of finding the *difference* between two like numbers. The *minuend* is the number to be diminished, and the *subtrahend* is the number to be taken from the minuend.

Addition and subtraction are closely related. Subtraction by adding is the method used by the expert cashier and by money changers. The "making change" method of subtraction consists in adding to the amount of the purchase enough to make the sum equal to the amount tendered in payment.

Example

Y buys groceries to the value of $1.34 and gives the cashier two one-dollar bills in payment. How much change should he receive?

Solution

The cashier in making change may return to Y a penny, a nickel, a dime, and a half dollar, saying: "$1.34, 35, 40, 50, $2.00," which means $1.34 + .01 = $1.35; $1.35 + .05 = $1.40; $1.40 + .10 = $1.50; and $1.50 + .50 = $2.00. Other coins than those mentioned may be returned by the cashier, but it is customary to make change in the largest coins possible.

Exercise

As the cashier, make change, using the largest denominations possible, assuming the following purchases were made and two one-dollar bills were offered in payment.

1. $1.44	**5.** $1.64	**9.** $1.17	**13.** $1.43
2. 1.67	**6.** 1.32	**10.** 1.29	**14.** 1.38
3. 1.27	**7.** 1.82	**11.** 1.54	**15.** 1.49
4. 1.41	**8.** 1.11	**12.** 1.56	**16.** 1.05

Avoid errors. Many errors in subtraction are made in borrowing from the next higher order. When that order is reached, it is not uncommon to overlook the fact that borrowing has taken place. Errors of this kind can be avoided by changing subtraction to the process of addition; that is, by adding to the subtrahend the number required to make the subtrahend equal to the minuend.

Explanation. Instead of thinking, "7 from 16 is 9," think, "7 + 9 = 16." Write the 9. Add 1, the digit carried over, to the 8, making 9. 9 + 8 = 17. Write 8, and add 1, the digit carried over, to 1, making 2. 2 + 0 = 2. Write 0. 3 + 5 = 8. Write 5. Answer: 5,089.

Example

Minuend	8276
Subtrahend......	3187
Difference........	5089

Problems

1. 9574	**2.** 7436	**3.** 6175	**4.** 8147	**5.** 6328	**6.** 5317
5886	3569	2897	4368	2549	3428

Difference between a given minuend and several subtrahends. In instances similar to the following example, the final result can be found in one operation by the application of the foregoing method of subtraction.

Example

From a fund of $3,456, the following disbursements were made: $594, $375, and $286. What was the balance left in the fund?

Explanation. Write the problem as shown in the solution. Begin at the right, and add the units' column of subtrahends, (6 + 5 + 4), adding (and setting down) enough (in this instance, 1) to make the units' figure of the sum the same as the units' figure of the minuend. Add the tens' column of the subtrahends, including the carrying figure, (1 + 8 + 7 + 9), adding (and setting down) enough (in this instance, 0) to make the tens' figure of the sum equal the tens' figure of the minuend. Add the hundreds' column of the subtrahends, including the carrying figure, (2 + 2 + 3 + 5), adding (and setting

Solution

$3,456
594
375
286
$2,201

down) enough (in this instance, 2) to make the hundreds' figure of the minuend. To the carrying figure, 1, add enough (in this case, 2) to make the thousands' figure of the minuend; set down 2.

Problems

1.	2.	3.	4.	5.
$1,562	$2,756 28	$5,987	$4,875	$2,975
437	52.70	235	365	762
122	7 55	789	1,529	194
254	528 75	1,526	284	275

Balancing an account. In most cases, inspection will tell which side of the account is the greater in amount. Add the larger side, and put the same footing on the smaller side, leaving space for the balance; then add from the top downward, supplying the figures necessary to make the column total equal to the footing previously placed there.

Example

Debits		*Credits*
$ 1,956.18		$ 134.26
3,452 75		258 19
289.34		764 83
5,726.31		2,375 94
	Balance,	7,891 36
$11,424 58		$11,424.58

Explanation. The balance, $7,891.36, was found as follows: Inspection showed the debit side to be the larger in amount. It was therefore added, and the footing of the account, $11,424.58, was placed under both debit and credit columns. The first order of the credits—that is, the cents—adds to 22. Insert 6 to make 28. With 2, the digit carried over, the second order, the dimes, adds to 22. Insert 3 to make 25. The third order, the dollars, with the digit carried over, adds to 23. Insert 1 to make 24. The fourth order, the tens of dollars, with the digit carried over, adds to 23. Insert 9 to make 32. The fifth order, the hundreds of dollars, with the digit carried over, adds to 16. Insert 8 to make 24. The sixth order, the thousands of dollars, with the digit carried over, adds to 4. Insert 7 to make 11.

Problems

1. *Debits*	*Credits*	2. *Debits*	*Credits*	3. *Debits*	*Credits*
$856.73	$298.56	$725 14	$1,356.17	$3,586.28	$ 591.18
345 96	264.39	239 51	691 35	192.75	2,751.26
298.85	6.15	64.28	256.38	384 72	185.35
142 31	....	....	75.19	265 54	....

Complement method. The complement of a number is the difference between that number and the unit of a next higher order Thus, the complement of 6 is 4; the complement of 8 is 2; and the complement of 68 is 32.

If, in subtracting a number less than 10 from a given number, its complement is added, the result will be 10 too large. If two

complements are added, the result will be 20 too large; and if three complements are added, the result will be 30 too large.

To find the sum of a column containing numbers to be subtracted, add the complements of the subtractive items, and from the sum of each order deduct as many tens as there are subtractive items in the order.

Example

A practical application of the complement method of subtraction is that of finding the net increase in a statistical record such as the following:

Dept.	*Sales This Mo.*	*Sales Last Mo.*	*Increase Decrease**
1	$ 427 95	$ 346 29	$ 81 66
2	515 86	457 75	58 11
3	395 57	385.86	9.71
4	402.75	416 87	14 12*
	$1,742.13	$1,606.77	$135.36

Solution

The difference between the sales this month and the sales last month for each department is shown as an increase or a decrease. The difference between the total sales this month and the total sales last month is $135.36. To prove that the departmental increases and decrease are correct, add the third column, beginning at the top and adding downward, using the complement each time on the last number. Thus, 8 and 8 are 16; write 6, and drop the 10, as one complement was added and the answer is 10 too large. 14 and 9 are 23; write 3 and carry 10, dropping one 10. 19 and 6 are 25; write 5 and carry 1, again dropping one 10. 14 and 9 are 23; write 13, dropping one 10 as before.

Example

Find the net increase of the following items:

*Increase Decrease**
15 60
4 51*
17 20
61 96
29 00
8 62*
124 20
59 40
89 83*
199.30
113 79*
132 46
34.99
122.65
580.01

Solution

In this problem there are four items showing decreases; therefore, each time a complement is added, the final result will be 10 too large, and in this case, the final result will be 40 too large, so 40 is deducted each time. Begin at the top and add downward: 9 (comp.), 15, 23 (comp. was 8), 30, 31, 37, 46, 51, subtract 40, write 1 and carry 1.

Now the next column. 7 (6 and 1), 12, 14, 23, 27, 29, 33, 35, 38, 41, 45, 54, 60, subtract 40, write 0 and carry 2. Next column, 7, 13, 20, 21, 30, 32, 36, 45, 46, 55, 62, 64, 68, 70, subtract 40, write 0 and carry 3.

Adding the tens: 4 (1 and 3 carried), 5, 11, 13, 15, 20, 22, 31, 40, 43, 46, 48, but subtract 20 as only two complements were used, write 8 and carry 2. The complement 10 may be

added each time there is no item, making the answer 68, then subtract 40, leaving 28 as before. Remember, subtract as many 10's as there are complements added.

Finally the hundreds' column. There are but five items in this column; therefore, with the 2 carried, proceed as follows: 3, 4, 13, 15, subtract 10 (only one complement was added) and write 5. Answer: 580.01.

Problems

The items to be subtracted are marked (*) in Problems 1 and 2.

1.	2.	3.	4.
$58 10	$122 65	$48.75 Gain	$20 25 Gain
19 66	175 50	31.25 Gain	4.50 Loss
45 55	89 88*	3.20 Loss	41 50 Gain
77 28	17 20	65.50 Gain	28 45 Gain
9 01*	1 48	15.25 Loss	38 47 Gain
16.11	8 62*	16.38 Gain	12 34 Loss
14 12*	36 95	26 65 Gain	49 82 Gain

Subtracting on an adding machine. If increase or decrease columns are being verified on an adding machine that does not have a direct subtraction device, add the complements of the numbers to be subtracted.

To subtract $219.48, set 780.52 on the keyboard and strike all nines to the left of the number; and to subtract $102.79, set 897.21 and strike all nines to the left of the number. Striking of the nines eliminates from the totalizers the number 1 that would otherwise be included in the answer.

Practical problems. In the following problems, both addition and subtraction have to be performed in order to complete the records.

Problem 1

This problem illustrates a section of a twelve-month moving-average schedule used in cost accounting and other cumulative work. Assuming that twelve months covers a cycle of business changes due to seasonal variations, and so forth, the moving twelve months' total provides a fairly reliable amount for comparative purposes.

The earliest month's results are subtracted from the twelve months' total and the current month's results are added, making a current twelve-month accumulation. The record is self-proving.

	Dept. 1	*Dept. 2*	*Dept. 3*	*Dept. 4*	*Total*
Total, 12/31/43..	$125,275.93	$56,472.29	$4,207.23	$7,200.49	
Deduct Jan., 1943	9,495.79	4,907.63	368.80	502.50	
Add Jan., 1944...	8,805.67	4,480.25	358.79	588.79	
12 mos. totals....					
Deduct Feb., 1943	8,933.07	4,093.19	293.67	496.68	
Add Feb., 1944...	9,033.48	4,123.97	235.80	517.90	
12 mos. totals....					
Deduct Mar., 1943	10,854.92	4,837.07	331.04	480.09	
Add Mar., 1944..	8,588.37	4,001.18	334.17	521.72	
12 mos. totals.					

Problem 2

From the following sales record, find the increase or decrease in sales by departments.

COMPARATIVE SALES RECORD

Dept. No.	*February, 2nd Year*	*February, 1st Year*	*Increase or Decrease*†
1	$ 7,134.95	$ 6,834.79	
2	6,225 19	5,764 87	
3	7,934 97	8,375 16	
4	6,354.76	5,986 35	
5	3,695.15	3,756 89	
6	9,767 98	9,475 18	
7	8,567 39	8,467.57	
8	5,607 18	4,865 84	
9	11,365 39	10,785 65	
10	14,572 86	13,764 16	
Total			

Problem 3

A daily business record may be prepared from cash register totals and other information. With the aid of the amounts given, complete the record for the day. Some of the sections contain items that are needed to complete other sections.

Cash Receipts		*Sales*		*Cash Paid Out*	
Rec'd. on Acc't.	$234 56	Cash Sales	$	For Stock	$ 85 42
Other Receipts	59 32	Credit Sales	152.35	For Expenses	19 56
Cash Sales	497.85			Personal	27.50
				Deposit	652.80
Total Receipts		Total Sales		Total	

Cash on Hand		*Bank Account*		*Accounts Receivable*	
Opening Balance	$250.75	Bal. for'd	$2,872.63	Bal. for'd	$481.52
Receipts		Today's Dep		Credit Sales	
Total		Total		Total	
Paid Out		Today's Cks.	175.32	Rec'd. on Acct.	
Closing balance		Balance		Balance	

Accounts Payable		*Cash Sales Summary*		*Credit Sales Summary*	
Bal. for'd	$315.20	Total for'd	$2,542.75	Total for'd	$638.47
Invoices Today	262.35	Today's Cash Sales		Today's Credit Sales	
Total		Total to forward		Total to forward	
Paid Today	136.57				
Balance					

Problem 4

In the following table of Gross Profits by Departments, add the Goods on Hand, March 1, 1st Year, to the Purchases for the Year, and from this sum subtract the Goods on Hand, March 1, 2nd Year. This gives the Cost of Goods Sold. The operation should be performed without transferring any of the figures. Use the complements of the numbers in the column Goods on Hand, March 1, 2nd Year.

The difference between the Cost of Goods Sold and the Sales will give the Profit or Loss.

To verify the work, add all the columns, and deal with the totals in the same way as with the figures for the departments. The difference between the total Cost of Goods Sold column and the Sales column should equal the difference between the Profit and the Loss columns, showing the Net Profit of the ten departments for the year.

GROSS PROFITS BY DEPARTMENTS

Dept.	*Goods On Hand March 1, 1st Year*	*Purchases for the Year*	*Goods On Hand March 1, 2nd Year*	*Cost of Goods Sold*	*Sales*	*Profit*	*Loss*
1	$3,475 86	$ 9,846 37	$2,347 11		$12,678 92		
2	1,357.10	6,725 40	1,475 86		6,188 90		
3	3,276 84	10,326 85	3,827 84		8,297 63		
4	5,475.90	11,176 98	5,874 13		13,586 47		
5	4,276 83	9,798 34	4,207 16		10,508 92		
6	3,785 47	8,376.41	3,648 10		8,756 13		
7	2,986 17	9,386 57	3,014 74		8,964 85		
8	3,275.83	8,724 18	2,817 56		9,575 34		
9	2,976 95	9,543 34	2,734 15		10,789 18		
10	3,532 25	10,217 60	3,375 89		12,756 84		
Footings							

Multiplication. Multiplication is a short process of addition; that is, a number is to be taken as an addend a given number of times.

How many bushels of grain are in three bins each containing 146 bu.?

Addition	*Multiplication*
146	146
146	3
146	438
438	

Multiplication involves three numbers, the multiplicand (the number to be repeated, 146); the multiplier (the number showing the number of repetitions, 3); and the product (the number showing the result, 438).

The multiplicand and the product are always like numbers. 146 bushels multiplied by 3 equals 438 bushels.

Problems

1. What is the cost of 640 acres of land at $42.50 an acre?

2. How many minutes are there in an ordinary year?

3. A barrel of flour contains 196 pounds. What is the weight of flour produced in one day by a mill that produces 375 barrels?

4. Sound travels about 1,120 feet in a second. How far will it travel in 15 seconds?

5. How many peaches are in 12 crates, if there are 84 peaches in each crate?

Accuracy and speed in multiplication depend largely upon a thorough mastery of the multiplication tables. Tables previously learned should be reviewed. Continue with frequent drills on combinations up to 25 × 25. The following table of multiples from 12 × 12 to 25 × 25 is given for reference and drill. Tables of multiples prepared in this manner facilitate the work of pay roll extension, inventory extension, billing, and so forth.

TABLE OF MULTIPLES

	12	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**
12	144	156	168	180	192	204	216	228	240	252	264	276	288	300
13	156	169	182	195	208	221	234	247	260	273	286	299	312	325
14	168	182	196	210	224	238	252	266	280	294	308	322	336	350
15	180	195	210	225	240	255	270	285	300	315	330	345	360	375
16	192	208	224	240	256	272	288	304	320	336	352	368	384	400
17	204	221	238	255	272	289	306	323	340	357	374	391	408	425
18	216	234	252	270	288	306	324	342	360	378	396	414	432	450
19	228	247	266	285	304	323	342	361	380	399	418	437	456	475
20	240	260	280	300	320	340	360	380	400	420	440	460	480	500
21	252	273	294	315	336	357	378	399	420	441	462	483	504	525
22	264	286	308	330	352	374	396	418	440	462	484	506	528	550
23	276	299	322	345	368	391	414	437	460	483	506	529	552	575
24	288	312	336	360	384	408	432	456	480	504	528	552	576	600
25	300	325	350	375	400	425	450	475	500	525	550	575	600	625

Contractions in multiplication. Contractions in multiplication may often be made by observing the peculiarities of the multiplier and the multiplicand and calling into use factors, multiples, complements, supplements, reciprocals, aliquots, and the like.

To multiply by factors of the multiplier. The ordinary method and the shorter method of multiplying by factors are shown in the following example. Observe that in the ordinary method there are two multiplications and an addition, while in the shorter method there are only two multiplications.

Example

Multiply 567 by 27.

Solution

Ordinary Method	*Shorter Method*
567	567 $27 = 9 \times 3$
27	9
3969	5103
1134	3
15309	15309

Problems

Multiply:

1. 4,584 by 64.	**3.** 1,459 by 35.	**5.** 8,756 by 42.
2. 8,359 by 54.	**4.** 2,684 by 27.	**6.** 6,123 by 45.

To multiply when a part of the multiplier is a factor or multiple of another part.

Example

Multiply 34,768 by 488.

Solution

```
 34768
   488
 ------
 278144   product by 8
16688640  product of 60 times product by 8
--------
16966784
```

Problems

Multiply:

1. 45,692 by 549.	**3.** 21,347 by 497.	**5.** 84,123 by 248.
2. 49,871 by 648.	**4.** 33,546 by 355.	**6.** 13,456 by 153.

To multiply a number of two figures by 11. Observation of the ordinary method shows that, in the answer, the sum of the two digits is written between the two digits.

Ordinary Method	*Shorter Method*
54	54
11	11
54	594
54	
594	

When the sum of the two digits is 10 or more, 1 must be carried to the digit at the left; for example, $64 \times 11 = 704$, and $93 \times 11 = 1{,}023$.

To multiply any number by 11. Observation of the ordinary method shows that, in the answer, the units' digit of the multiplicand is the units' digit of the product; that the tens' digit of the product is the sum of the units' digit and the tens' digit of the multiplicand; that the hundreds' digit of the product is the sum of the tens' digit and the hundreds' digit of the multiplicand; and so on. When the sum of two digits is 10 or more, 1 must be carried.

Ordinary Method	*Shorter Method*
8937	8937
11	11
8937	98307
8937	
98307	

Multiplying by 25. Annex two ciphers to the multiplicand, and divide by 4.

Example

Multiply 7,562 by 25.

Solution

$$4\overline{)756200}$$
$$189050$$

Problems

Multiply each of the following by 25:

1. 3,874. 2. 3,948. 3. 7,981. 4. 5,426.

Multiplying by 15. Annex a cipher to the multiplicand, and increase the result by one-half of the multiplicand.

Example

Multiply 8,435 by 15.

Solution

$$\begin{array}{r} 84350 \\ 42175 \\ \hline 126525 \end{array}$$

Problems

Multiply each of the following by 15:

1. 7,432. 2. 8,397. 3. 3,926. 4. 9,536.

Multiplying numbers ending with ciphers. Multiply the significant figures in each number, and to the product annex as many ciphers as there are final ciphers in both the multiplier and the multiplicand.

Example

Multiply 756,000 by 4,200.

Solution

$$\begin{array}{r} 756 \\ 42 \\ \hline 31752 \end{array}$$

Annex five ciphers. Answer: 3,175,200,000.

Problems

Multiply:

1. 325,000 by 2,300.
2. 370 by 480.
3. 24,100 by 4,200.
4. 8,300 by 2,100.

Multiplication by numbers near 100, as 98, 97, 96, and so forth, and by numbers near 1,000, as 997, 996, and so forth. This method is of value in finding the net proceeds of some amount less 2%, 3%, and so forth, and also in many other situations.

Example

Multiply 3,247 by 97.

Solution

Multiply the number by 100, and subtract 3 times the number.

$$\begin{array}{r l}
324{,}700 & = 3{,}247 \times 100 \\
\underline{9{,}741} & = 3{,}247 \times 3 \\
314{,}959 & = 3{,}247 \times 97
\end{array}$$

Multiplication by a number near 1,000 is accomplished in the same manner by multiplying by 1,000 instead of by 100.

Problems

Multiply:

1. 2,459 by 98. **2.** 7,318 by 97. **3.** 5,438 by 96. **4.** 8,752 by 95.

Multiplication of two numbers each near 100, 1,000, and so forth. Products of numbers in this class may be calculated mentally.

Example

Multiply 96 by 98.

Explanation. Step 1. Multiply the complements of the two numbers, and if the product occupies units' place only, prefix a cipher. Result, 08.

Step 2. Subtract the complement of one number from the other number, and write the result at the left of the result in Step 1. The complement of either number subtracted from the other number leaves the same remainder; as, 96 − 2 or 98 − 4 each equals 94. Answer: 9,408.

Solution

	Complement
96	4
98	2
9408	

Example

Multiply 92 by 88.

Solution

	Complement
92	8
88	12
8096	

Explanation. The product of the complements is 96, the last two figures of the answer. 88 − 8 or 92 − 12 = 80, the first two figures of the answer. Answer: 8,096.

Example

Multiply 996 by 988.

Solution

	Complement
996	4
988	12
984,048	

Explanation. When numbers near 1,000 are multiplied, ciphers are prefixed to the product of the complements, so that the product occupies three places.

Problems

Multiply:

1. 97 by 96. **2.** 88 by 98. **3.** 995 by 992. **4.** 997 by 994.

Multiplying by numbers a little larger than 100, as 101, 102, and so forth. Annex two ciphers to the multiplicand, and to this add the product of the multiplicand and the units' figure of the multiplier. Annex three ciphers for multipliers over 1,000.

Example

Multiply 3,475 by 104.

Solution

```
347500
 13900   (4 × 3,475)
------
361400
```

Problems

Multiply:

1. 2,875 by 102. **2.** 3,496 by 105. **3.** 2,972 by 1,004. **4.** 4,568 by 1,006.

Multiplication of two numbers each a little more than 100. To the sum of the numbers (omitting one digit in the hundreds' column), annex two ciphers, and add the product of the supplements (excess over 100).

Example

Multiply 112 by 113.

Solution

```
112
113
12500   (sum of numbers, with one digit in the hundreds' column omitted)
  156   (product of supplements, 12 × 13)
-----
12656
```

Explanation. In instances similar to the foregoing, a knowledge of the multiplication tables to 20 × 20 makes mental results possible, and is invaluable in inventory and other extensions.

Problems

Multiply:

1. 114 by 112. **2.** 106 by 108. **3.** 116 by 111. **4.** 118 by 115.

Cross multiplication. When the multiplicand and the multiplier are each numbers of two figures, the work may easily be kept in mind and the partial products added without being written down.

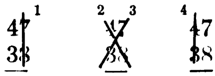
Example

Multiply 47 by 38.

Solution

```
  47
  38
----
1786
```

Graphic Solution



Explanation. $8 \times 7 = 56$. Write 6, carry 5. $(8 \times 4) + (3 \times 7) + 5 = 58$. Write 8, carry 5. $(3 \times 4) + 5 = 17$. Write 17. Answer: 1,786.

Problems

Multiply:

1. 53 by 29. **2.** 48 by 57. **3.** 74 by 32. **4.** 65 by 28

To cross-multiply a number of three digits by a number of two digits. A three-digit number may be multiplied by a two-digit number in a manner similar to that of multiplying a two-digit number by a two-digit number.

Example

Multiply 346 by 28.

Solution

```
 346
  28
----
9688
```

Explanation. $8 \times 6 = 48$. Write 8, carry 4. 4 (carried) + (8×4) + $(6 \times 2) = 48$. Write 8, carry 4. 4 (carried) + $(8 \times 3) + (4 \times 2) = 36$. Write 6, carry 3. 3 (carried) + $(2 \times 3) = 9$. Write 9. Answer: 9,688.

A graphic presentation of the steps required appears as follows:

Problems

1. 324×28	**4.** 428×34	**7.** 289×85	**10.** 693×42
2. 543×42	**5.** 516×26	**8.** 356×48	**11.** 384×56
3. 658×56	**6.** 513×76	**9.** 785×34	**12.** 473×65

To cross-multiply a number of three digits by another number of three digits. Comparison of the graphic presentation with that above shows that the first three steps are the same, the next three are new, and the final three are the same.

Example

Multiply 428 by 356.

Solution

```
  4 2 8
  3 5 6
-------
152,368
```

Graphic Solution

Explanation. 6 × 8 = 48. Write 8, carry 4. 4 (carried) + (6 × 2) + (8 × 5) = 56. Write 6, carry 5. 5 (carried) + (6 × 4) + (8 × 3) + (2 × 5) = 63. Write 3, carry 6. 6 (carried) + (5 × 4) + (2 × 3) = 32. Write 2, carry 3. 3 (carried) + (3 × 4) = 15. Write 15. Answer: 152,368.

Problems

1. 124 × 251	**6.** 832 × 425	**11.** 436 × 579
2. 262 × 158	**7.** 639 × 256	**12.** 832 × 656
3. 328 × 245	**8.** 819 × 325	**13.** 295 × 638
4. 638 × 256	**9.** 677 × 283	**14.** 767 × 842
5. 784 × 364	**10.** 518 × 824	**15.** 698 × 476

Preparation of a table of multiples of a number. It is not uncommon to have to use the same number many times in making calculations, especially in cost accounting. A saving of time and increased accuracy in the work are achieved if a table of multiples of the number is constructed. Suppose that you have to perform a number of multiplications in which 326,834 is one of the factors. A table of multiples may be constructed with an adding machine by locking the repeat key. Sub-total after each pull of the handle. The sub-totals should check with the product column shown below. If the table is prepared by repeated additions, and not with an adding machine, the 10th product should be computed, as it will verify all, unless there are compensating errors in the work.

TABLE OF MULTIPLES

Multiplier	*Product*
1	326,834
2 (326,834 + 326,834)	653,668
3 (653,668 + 326,834)	980,502
4 (980,502 + 326,834)	1,307,336
5 (1,307,336 + 326,834)	1,634,170
6 (1,634,170 + 326,834)	1,961,004
7 (1,961,004 + 326,834)	2,287,838
8 (2,287,838 + 326,834)	2,614,672
9 (2,614,672 + 326,834)	2,941,506
Verification	
10 (2,941,506 + 326,834)	3,268,340

Example

Multiply 326,834 by 5,249.

Solution

2941506	= 9 times 326,834
1307336	= 4 times 326,834
653668	= 2 times 326,834
1634170	= 5 times 326,834
1715551666	= product

If the table is prepared without the use of an adding machine, proceed as outlined on the next page.

1. Write 326,834 near the bottom of a slip of paper or a card.

2. Start the table by writing 326,834. Place the slip or card just above this number, thus:

```
      ┌─────────┐
      │ 326,834 │
      └─────────┘
  1.    326,834
  2.    ........
  3.    ........
```

3. Add the two numbers, placing the sum, 653,668, on line 2. This is two times the number.

4. Move the slip or card down one line and add again, placing the sum, 980,502, on line 3, forming three times the number.

5. Continue moving the slip or card down one line each time and adding.

6. When 9 times the number is obtained, check the accuracy of the work by repeating the process once more. The result should be ten times the number.

Problems

Set up a table of multiples of 245,386, and multiply 245,386 by the following numbers:

1. 2,465 **2.** 3,542 **3.** 2,498 **4.** 5,347 **5.** 6,173

Division. Division is the process of finding how many times one number is contained in another number. The *dividend* is the number to be divided, the *divisor* is the number by which we divide, and the *quotient* is the number showing how many times the dividend contains the divisor.

The *remainder* is a number less than the divisor, and results when the dividend does not contain the divisor exactly. It is an undivided portion of the dividend.

Short division is the method used when the products of the divisor and the digits of the quotient are omitted.

Example	*Solution*
Divide 3,476 by 2.	2)3476 1738

Long division is the method used when the work is written in full.

Example

Divide 5,839 by 24.

Solution

```
24)5839(243
   48
   ---
   103
    96
   ---
     79
     72
     --
      7
```

To divide by 25, 50, or 125. The work of division can be lessened by making the operation one of multiplication.

Example	*Solution*
Divide 1,400 by 25.	$14 \times 4 = 56.$

Explanation. Divide 1,400 by 100 by dropping the zeros. But, 100 is 4 times the actual divisor, therefore the quotient 14 is $\frac{1}{4}$ of the actual quotient, so 14×4 or 56 is the actual quotient.

In a similar manner, 1,400 divided by 50 is 28; and 14,000 divided by 125 is 112. (*Note:* Further reference to this method is given under the subject of division by aliquot parts of 100.)

Abbreviated division. Instead of writing the product and then subtracting, the product of each digit of the divisor is subtracted mentally, using the "making change" method, and only the remainder is written.

```
        3285
  234)768756
      667
      1995
       1236
         66
```

Use of tables in division. If a number of divisions are to be made with the same divisor, it is advantageous to set up a table of multiples of the divisor.

Example

Assume that 328 is to be used a number of times as a divisor, and that one of the dividends is 587,954, a table of multiples could be set up thus:

TABLE OF MULTIPLES

Multiplier	*Product*
1	328
2	656
3	984
4	1,312
5	1,640
6	1,968
7	2,296
8	2,624
9	2,952

Explanation. Inspection shows the first digit in the quotient to be 1. The second partial dividend is 2,599. The table of multiples shows the largest product contained therein to be 2,296, opposite 7. The third partial dividend is 3,035, and the table of multiples shows the largest product contained therein to be 2,952, opposite 9. The fourth partial dividend is 834, and the largest product contained therein is 656, opposite 2. The remainder is 178. The fraction $\frac{178}{328}$ may be reduced to $\frac{89}{164}$, or it may be changed to a decimal.

Solution

```
328)587954(1792 89/164
    328
    2599
    2296
     3035
     2952
       834
       656
       178
```

$$\frac{178}{328} = \frac{89}{164}$$

Division in this manner is rapid, as no time is lost through selection of a quotient so large that when the product is found it exceeds the dividend, necessitating another trial.

Problems

Divide the following numbers by 144 after setting up a table of multiples of 144:

1. 374,825. **2.** 628,256. **3.** 496,287.

Reciprocals in division. The reciprocal of any number is found by dividing 1 by the number. The reciprocal of 5 is 1 ÷ 5, or .2, and the reciprocal of 25 is 1 ÷ 25, or .04.

The quotient in a division may be found by multiplying the dividend by the reciprocal of the divisor. Hence, in instances in which it is necessary to find what per cent each item is of the total of the items, the use of the reciprocal of the divisor will save time and provide a check on these computations.

To find what per cent each item is of the total of the items:

(*a*) Divide 1 by the total of the items to obtain the reciprocal of the total.

(*b*) Using the result obtained in (*a*) as a fixed multiplier, multiply each of the individual items, and the respective results obtained will be the per cents which the individual items are of the total sum.

Example

Find the per cent that each department's monthly expense is of the total monthly expense.

Department	*Expense*
A	$ 600 00
B	500 00
C	1,200 00
D	700 00
E	1,000 00
Total	$4,000 00

Solution

Divide 1 by 4,000 to obtain the reciprocal, .00025. Multiply the expense of each department by this reciprocal, and the product will be the per cent that the department's expense is of the total expense.

Department	*Expense*		*Reciprocal*		*Per Cent*
A	$ 600 00	×	.00025	=	15 %
B	500.00	×	.00025	=	12½%
C	1,200 00	×	00025	=	30 %
D	700 00	×	.00025	=	17½%
E	1,000 00	×	.00025	=	25 %
Total	$4,000 00				100 %

The foregoing method of calculating the rate per cent has a great many applications in an accountant's work. Another illustration is given—that of calculating the per cent that each item in a profit and loss statement is of net sales.

QUALITY MEAT MARKET

PROFIT AND LOSS STATEMENT FOR THE YEAR

	Detail	*Amount*	*Per Cent*
Net sales		$20,000 00	100 00
Cost of merchandise sold		15,712.00	78.56
Gross profit		$ 4,288.00	21.44
Expenses			
Salaries and wages	$2,266 00		11.33
Advertising	22 00		.11
Wrappings	172 00		.86
Refrigeration	210 00		1.05
Heat, light, and power	54 00		.27
Telephone	54 00		.27
Rent	338 00		1.69
Interest	146 00		.73
Depreciation of store equipment	152 00		.76
Repairs to store equipment	44 00		.22
Insurance	10 00		.05
Taxes	42 00		.21
Losses from bad debts	38 00		.19
Other expenses	284 00		1 42
Total expenses		3,832 00	19.16
Net profit		$ 456 00	2 28

Explanation. The foregoing is a simple statement, and the per cents can be determined mentally if each item is divided by the amount of net sales. For the purpose of illustration, however, find the reciprocal of $20,000.00, which is .00005 (1 ÷ 20,000); then multiply each item by this reciprocal, and the results will be as shown in the per cent column.

Problems

1. The floor space occupied by *Z* Manufacturing Company was as follows:

Service Department *X*	600 sq. ft.
Service Department *Y*	1,100 sq. ft.
Service Department *Z*	550 sq. ft.
Producing Department *A*	2,000 sq. ft.
Producing Department *B*	1,568 sq. ft.
Producing Department *C*	2,234 sq. ft.
Sales Department	600 sq. ft.
Administrative Offices	550 sq. ft.
	9,202 sq. ft.

The Building and Maintenance Expense account shows a total of $2,982.50. What amount of this expense should be distributed to each of the departments?

2. In the following tabulation, find the per cent that each department's floor space is of the total floor space:

	Sq. Ft. Floor Space	*Per Cent of Total*
Dept. 1	2,456	....
Dept. 2	1,014	....
Dept. 3	875	....
Dept. 4	1,252	....
Dept. 5	748	....
	6,345	100 00

3. Calculate the per cent that each item is of net sales.

THE FOOD MART

PROFIT AND LOSS STATEMENT

Net Sales		$35,600	100 00%
Cost of Merchandise Sold		27,969	
Gross Profit		$ 7,631	....
Expenses			
Salaries and Wages	$4,080		....
Advertising	28		....
Wrappings	266		....
Refrigeration	308		....
Heat, Light, and Power	106		....
Telephone	81		....
Rent	416		....
Interest	203		....
Depreciation of Store Equipment	147		....
Repairs to Store Equipment	45		....
Insurance	21		....
Taxes	39		....
Losses from Bad Debts	119		....
Other Expenses	490		....
Total Expenses		6,349	....
Net Profit		$ 1,282	....

CHAPTER 2

Checking Computations

Methods. Addition may be checked by adding the second time, adding from the bottom to the top if the first addition was from the top to the bottom. This is preferable to performing the work in the same way the second time, as a mistake once made is likely to be repeated.

Subtraction may be checked by adding the subtrahend and the remainder. The sum should equal the minuend.

Multiplication may be checked by interchanging the multiplier and the multiplicand and multiplying again.

Division may be checked by multiplying the divisor and the quotient, adding to this product any remainder. The answer should equal the dividend.

Rough check. Rough check is an approximate check and is often used to locate large errors. It is also used in determining approximate results. It is especially useful in checking misplacement of the decimal point in multiplication and division of decimal fractions.

A rough check of addition may be made as follows:

Example	*Check*
54,892	55
36,071	36
53,784	54
21,342	21
76,854	77
242,943	243

If the required result is thousands, disregard the three columns at the right, except to increase the fourth column sum by one if the digit in the third column is 5 or more. The check shows the answer to be approximately 243,000.

Absolute check. There is no such thing as an absolute check because there are always possibilities of offsetting errors, but the use of several methods of checking computations makes the probability of error so slight that one may rely on the result as correct.

Check numbers obtained by casting out the nines. A simple and easily remembered check is that of casting out the nines. Add

the digits of the number, divide the sum by nine, and use the remainder, which is called "the excess," as the check number. In the number 4,875, the sum of the digits is 24, and 24 divided by 9 equals 2 with an excess of 6.

Verification of addition.

Explanation. The sum of the digits of 8,342 is 17 (8 + 3 + 4 + 2). Cast out 9 and set down 8. If a number contains a 9, skip it in adding the digits; thus, in 8,967, 8 + 6 + 7 equals 21. Cast out the nines and set down the excess, 3. Find the check number of each line in the same way. Add the check numbers, and cast the nines out of their sum. Find the check number of the sum of the column being verified. The final check number in each case is 5.

Example

8342	8
8967	3
8378	8
9276	6
8431	7
43394—5	32—5

Problems

Add, and verify by casting out the nines:

1.	**2.**	**3.**	**4.**
2487	7452	4501	1231
3156	8129	2765	4567
2982	5758	4567	1085
4756	2253	8256	3426
8928	7685	2435	7531

✓7

Verification of subtraction.

Example

7856	8
2138	5
5718	3

Explanation. 7,856 checks 8, and 2,138 checks 5. 8 − 5 = 3, and 5,718 checks 3.

Problems

Subtract, and verify by casting out the nines:

1.	**2.**	**3.**	**4.**
7496	7428	4751	8237
2831	1956	3286	5129

Verification of multiplication.

Example

482	5
376	7
181232—8	35—8

Explanation. 482 checks 5, and 376 checks 7. 7 × 5 = 35. 35 checks 8, and the product, 181,232, also checks 8.

Problems

Multiply, and verify by casting out the nines:

1.	2.	3.	4.
456	412	832	765
287	654	254	414

Verification of division. Division may be verified by multiplication; that is, the product of the quotient and the divisor should equal the dividend. Apply the same principle in verifying with check numbers.

Example

```
13)76492(5884
   65
   114
   104
    109
    104
      52
      52
```

Explanation. 76,492 checks 1. 13 checks 4. 5,884 checks 7. $4 \times 7 = 28$, and 28 checks 1, which is also the check number of the dividend.

Problems

Divide, and verify by casting out the nines:

1. 11,550 by 42. **2.** 60,882 by 73. **3.** 11,049 by 127. **4.** 9,854 by 26.

Verification of division where there is a remainder. The check number of the remainder added to the product of the check number of the quotient and the check number of the divisor should equal the check number of the dividend.

Example

```
32)75892(2371
   64
   118
    96
    229
    224
      52
      32
```

Explanation. Step 1: The remainder, 20, checks 2. The quotient, 2,371, checks 4. The divisor, 32, checks 5. $2 + (4 \times 5) = 22$, and 22 checks 4.

Step 2: The dividend, 75,892, checks 4.

Step 1 and Step 2 should produce the same check number.

Problems

Divide, and verify by casting out the nines:

1. 34,765 by 52. **2.** 29,878 by 87. **3.** 95,763 by 26. **4.** 8,476 by 41

Check numbers obtained by casting out the elevens. Because casting out nines does not reveal errors in computations if two

digits have been transposed, some persons prefer to use eleven as a check number.

Begin with the left-hand digit of the first number, and subtract it from the digit to its immediate right. If the digit to the right is smaller, add eleven before subtracting. Using the remainder as a new digit, subtract it from the third digit from the left, first adding eleven if necessary. Use this remainder as a new digit, and subtract it from the fourth digit from the left, first adding eleven if necessary. Continue in this manner until all the digits in the number have been used. The final remainder is the check number of the number.

Another method of checking results by means of the number eleven is to use alternate digits. From the sum of the first, third, fifth, etc., digits (beginning at units' place) subtract the sum of the second, fourth, sixth, etc., digits. If the subtraction cannot be performed, eleven is first added to the sum of the odd digits, and the sum of the even digits is subtracted, the remainder being the check number.

Verification of addition.

Explanation. Begin at the left with the number 4,324. 4 from 14 (3 + 11) = 10. 10 from 13 (2 + 11) = 3. 3 from 4 = 1, the check number of 4,324.

Take the second number, 8,689. 8 from 17 (6 + 11) = 9. 9 from 19 (8 + 11) = 10. 10 from 20 (9 + 11) = 10, the check number of 8,689.

Check all the numbers in the same manner. Add the check numbers. The sum of the check numbers checks 1, and the sum of the numbers checks 1.

Example

4324	1
8689	10
6327	2
8964	10
3487	0
31791—1	23—1

Problems

Add, and verify by casting out the elevens:

1.	**2.**	**3.**	**4.**
3789	2456	9755	8307
5462	1279	8256	7165
9581	2075	3851	2693
3998	2754	8632	2198
5314	9287	6311	5183

Verification of subtraction.

Example

7453	6
1289	2
6164	4

Explanation. 7,453 checks 6. 1,289 checks 2. 6 − 2 = 4 and 6,164 checks 4.

Problems

Subtract, and verify by casting out the elevens:

1.	2.	3.	4.
8795	3465	7985	3079
1560	2134	5698	1002

Verification of multiplication.

Example

$$\begin{array}{r|l} 584 & 1 \\ \underline{256} & \underline{3} \\ 149504 & 3 \end{array}$$

Explanation. 584 checks 1. 256 checks 3. $3 \times 1 = 3$, and 149,504 checks 3

Problems

Multiply, and verify by casting out the elevens:

1.	2.	3.	4.
346	4289	7437	287
275	324	2856	36

Verification of division.

Example 1	*Example 2*
24)89784(3741	31)75893(2448
72	62
177	138
168	124
98	149
96	124
24	253
24	248
	5

Explanation 1. 89,784 checks 2. 24 checks 2. 3,741 checks 1. $2 \times 1 = 2$, the check number of the dividend.

Explanation 2. 75,893 checks 4. 31 checks 9. 2,448 checks 6. The remainder checks 5. $5 + (9 \times 6) = 59$. 59 checks 4, the same check number as that of the dividend.

Problems

Divide, and verify by casting out the elevens:

1. 80,925 by 83. **2.** 124,392 by 142. **3.** 25,874 by 49. **4.** 28,769 by 135.

Check number thirteen. If thirteen is used as a check number, transpositions and shiftings of figures are readily detected. However, in checking by 13, it is necessary actually to divide by 13.

TABLE OF MULTIPLES

1	13	6	78
2	26	7	91
3	39	8	104
4	52	9	117
5	65	10	130

All the dividing is done mentally.

Example

Cast out 13 from 247,563.

Explanation. Begin with the two left-hand digits. 24 checks 11. 11, with the next digit, 7, is 117, and 117 checks 0. Use the next two digits. 56 checks 4. 4 with the next digit is 43, and 43 checks 4.

The verification of addition, subtraction, multiplication, and division is performed in the same manner as with 9 and 11. The difference is in the method of arriving at the check number, as has been outlined.

Problems

1. Add, and verify by check number 13:

```
24875
32986
79840
80475
13048
93476
```

2. Subtract, and verify by check number 13:

```
84756
21348
```

3. Multiply, and verify by check number 13:

```
4875
 259
```

4. Divide, and verify by check number 13:

$$\frac{975{,}648}{348}$$

CHAPTER 3

Factors and Multiples

Factors. The *factors* of a number are the integers whose product is the number. Thus, the factors of 6 are 2 and 3, the factors of 18 are 3 and 6, or 2 and 9. A prime factor is a prime number, that is, a number not exactly divisible by any number except itself and 1.

Factoring is the process of separating a number into its factors.

Example		*Solution*
What are the prime factors of 315?	3)315 3)105 5) 35 7	The prime factors of 315 are, therefore, $3 \times 3 \times 5 \times 7$.

Example		*Solution*
What are the factors of 315?	9)315 7) 35 5	The factors of 315 are, therefore, $9 \times 7 \times 5$.

Factoring is important for its assistance in the solution of problems in fractions, practical measurements, percentage, and all problems in which cancellation is used. One use of factors was given on page 17, "to multiply by factors of the multiplier," and another on page 18, "to multiply when a part of the multiplier is a factor or multiple of another part."

Tests of divisibility. To be able to factor a number quickly, one must become thoroughly familiar with the tests of divisibility.

A number is divisible by:

1. Two, if is is an even number or if it ends in zero.
2. Three, if the sum of its digits is divisible by 3. Thus, 41754 is divisible by 3 because the sum of the digits is 21, and 21 is divisible by 3.
3. Four, if the two right-hand figures are zeros, or if they express a number divisible by 4. Thus, 13724 is divisible by 4 because 24 is divisible by 4.
4. Five, if the units' figure is either a zero or a 5.
5. Six, if it is an even number the sum of whose digits is divisible by 3. Thus, 846, 918, and 54252 are divisible by 6.

6. Eight, if the three right-hand digits are zeros, or if they express a number divisible by 8. Thus, 2000 and 5624 are divisible by 8.

7. Nine, if the sum of its digits is divisible by 9.

8. Ten, if the right-hand figure is zero.

(There is no simple method of testing divisibility by 7.)

Greatest common divisor. A common divisor of two or more numbers is a number that evenly divides each of them. Thus, a common divisor of 16 and 24 is 4.

The greatest common divisor of two or more numbers is the greatest number that will evenly divide each of them. It is the product of all their common factors.

Example	*Solution*
Find the greatest common divisor of 36, 63, and 54.	$\begin{array}{r\|lll} 3 & 36 & 63 & 54 \\ \hline 3 & 12 & 21 & 18 \\ \hline & 4 & 7 & 6 \end{array}$

Since 4, 7, and 6 have no common factors, the G. C. D. is $3 \times 3 = 9$.

A practical application of the principles involved in finding the G. C. D. is in reducing common fractions to their lowest terms.

Problems

Find the G. C. D. of the following:

1. 64, 160, 320, 640

2. 36, 54, 90

3. 32, 48, 128

4. 81, 729, 2187

5. X, Y, and Z own land on a new street. X has 600 feet frontage, Y has 720 feet, and Z has 900 feet. If they wish to cut this land into lots of equal width, how wide will the lots be, and how many will each have?

6. If you have three coils of steel cable measuring, respectively, 2205, 2940, and 4704 feet, and wish to cut the whole quantity into pieces of the greatest equal length possible without waste or splices, what will be the length of each piece? How many lengths will be cut from each coil?

Least common multiple. A common multiple of two or more numbers is a number that is evenly divisible by each of them. Thus, 24 is a common multiple of 3 and 8.

The least common multiple of two or more numbers is the least number that is evenly divisible by each of them. Thus, 12 is the L. C. M. of 4 and 6.

Example

What is the L. C. M. of 12, 28, 30, 42, and 64?

Solution

$$\begin{array}{r|rrrrr}
2 & 12 & 28 & 30 & 42 & 64 \\ \hline
2 & 6 & 14 & 15 & 21 & 32 \\ \hline
3 & 3 & 7 & 15 & 21 & 16 \\ \hline
7 & 1 & 7 & 5 & 7 & 16 \\ \hline
 & 1 & 1 & 5 & 1 & 16
\end{array}$$

$$2 \times 2 \times 3 \times 7 \times 5 \times 16 = 6{,}720$$

Explanation. Notice that any number not divisible by the factor is brought down, and the process is repeated as long as at least two of the numbers have a common factor. Finally, the L. C. M. is the product of the factors and the numbers having no common factor.

Problems

Find the L. C. M. of the following:

1. 6, 18, 30, 42

2. 16, 24, 64, 96

3. 45, 63, 72, 99

4. 14, 35, 42, 28

Cancellation. Certain computations involving division can be shortened by removing or cancelling equal factors from both dividend and divisor.

Example

If 32 units of product sell for $57.60, what will 18 units of the same product sell for at the same rate?

Solution

$$\frac{\overset{9}{\cancel{18}} \times \overset{\overset{3.60}{\cancel{14.40}}}{\cancel{57.60}}}{\underset{\underset{\cancel{4}}{\cancel{16}}}{\cancel{32}}} = 32.40$$

Problems

Using cancellation, divide:

1. $\dfrac{27 \times 48 \times 96 \times 38}{19 \times 16 \times 9 \times 2}$

2. $\dfrac{8 \times 12 \times 15 \times 6}{5 \times 4 \times 3 \times 18}$

3. If 15 tons of coal cost $258.00, how much will 25 tons cost at the same rate?

4. A ship's provisions will last 36 men for 216 days. How long will they last 124 men?

CHAPTER 4

Common Fractions

Terms explained. A *unit* is a single quantity by which another quantity of the same kind is measured: 1 foot is the unit of 5 feet; 1 barrel is the unit of 10 barrels; 1 acre is the unit of 40 acres, and so forth.

These integral units are often divided into equal parts known as *fractional units*, as $\frac{1}{2}$ ft., $\frac{1}{4}$ bbl., $\frac{1}{3}$ A., and so forth.

A *fraction* is an expression for one or more of the equal parts of a unit, as $\frac{1}{2}$ ft., $\frac{3}{4}$ ft., $\frac{2}{3}$ bbl., $\frac{5}{6}$ A., and so forth.

The number above the line in the expression of a fraction is called the *numerator*; the number below the line is called the *denominator*.

The *denominator* indicates the number (and hence the size) of parts into which the unit is divided.

The *numerator* indicates the number of these parts taken.

A *proper fraction* expresses less than a unit, or its numerator is less than its denominator; as, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, and so forth.

An *improper fraction* is a fraction whose numerator is equal to or greater than its denominator; as, $\frac{3}{3}$, $\frac{5}{4}$, $\frac{9}{8}$, and so forth.

A *mixed number* is a number expressed by a whole number and a fraction; as, $2\frac{1}{4}$, $3\frac{1}{2}$, $16\frac{3}{4}$, and so forth.

Reduction of fractions. Reduction is the process of changing the numerator and the denominator of a fraction without changing the value of the fraction.

A fraction is reduced to *higher terms* when the numerator and the denominator are expressed in larger numbers.

A fraction is reduced to *lower terms* when the numerator and the denominator are expressed in smaller numbers, and it is reduced to its *lowest terms* when there is no common divisor of its numerator and denominator.

Principle. Multiplying or dividing both numerator and denominator of a fraction by the same number does not change the value of the fraction. Thus, $\frac{16}{24}$ may be reduced to the equivalent fraction $\frac{4}{6}$ by dividing both terms by 4. The fraction $\frac{16}{24}$ has been reduced to lower terms. Again, $\frac{16}{24}$ may be reduced to the equivalent fraction $\frac{2}{3}$ by dividing both terms by 8. Here the

fraction $\frac{16}{24}$ has been reduced to lowest terms, since 2 and 3 do not have a common divisor.

Conversely, $\frac{2}{3}$ may be changed to an equivalent fraction whose denominator is 24 by multiplying both terms by 8 (obtained by dividing 24 by 3), or $\frac{16}{24}$. Thus, the fraction $\frac{2}{3}$ has been reduced to a higher given denominator.

Mixed numbers. It is sometimes desirable to change a mixed number to an improper fraction, or, conversely, to change an improper fraction to a mixed number.

To change a mixed number to an improper fraction. Multiply the whole number by the denominator of the fraction, add the numerator, and place the sum over the denominator, thus, $3\frac{1}{3}$ is $\frac{10}{3}$, $4\frac{2}{5}$ is $\frac{22}{5}$, and $6\frac{1}{3}$ is $\frac{19}{3}$.

To change an improper fraction to a whole or a mixed number, divide the numerator by the denominator; thus $\frac{12}{3}$ is 4, $\frac{6}{5}$ is $1\frac{1}{5}$, $\frac{12}{9}$ is $1\frac{3}{9}$ or $1\frac{1}{3}$, and $\frac{19}{4}$ is $4\frac{3}{4}$.

Problems

1. Reduce to lowest terms: $\frac{8}{16}$, $\frac{6}{24}$, $\frac{4}{12}$, $\frac{16}{24}$, $\frac{18}{30}$, $\frac{72}{96}$, $\frac{42}{49}$, $\frac{36}{60}$, $\frac{13}{39}$, $\frac{25}{30}$.

2. Change to equivalent fractions having denominators as indicated:

$\frac{1}{2}$ to 8ths	$\frac{1}{5}$ to 15ths	$\frac{4}{5}$ to 25ths
$\frac{2}{3}$ to 6ths	$\frac{1}{6}$ to 24ths	$\frac{5}{16}$ to 48ths
$\frac{4}{5}$ to 20ths	$\frac{3}{8}$ to 24ths	$\frac{3}{8}$ to 32nds
$\frac{1}{4}$ to 8ths	$\frac{5}{9}$ to 36ths	$\frac{7}{12}$ to 36ths.

3. Reduce to equivalent fractions whose denominators are 24: $\frac{1}{12}$, $\frac{2}{3}$, $\frac{5}{6}$, $\frac{3}{4}$, $\frac{7}{8}$, $\frac{7}{6}$.

4. Change to improper fractions: $4\frac{1}{8}$, $3\frac{1}{3}$, $1\frac{1}{2}$, $7\frac{1}{4}$, $8\frac{2}{3}$, $6\frac{1}{4}$, $3\frac{2}{3}$, $5\frac{3}{4}$, $5\frac{5}{9}$, $9\frac{4}{5}$.

5. Change to whole or mixed numbers: $\frac{48}{8}$, $\frac{12}{7}$, $\frac{32}{6}$, $\frac{20}{8}$, $\frac{72}{12}$, $\frac{16}{3}$, $\frac{8}{5}$, $\frac{64}{7}$, $\frac{96}{11}$, $\frac{17}{3}$.

6. Is the number of fractional units increased or decreased when we reduce $\frac{9}{12}$ to $\frac{3}{4}$? Is the size of the fractional unit increased or decreased when we reduce $\frac{9}{12}$ to $\frac{3}{4}$?

Addition and subtraction of fractions. Similar fractions are fractions that have a common denominator. Only similar fractions can be added or subtracted.

To add fractions, reduce the fractions to similar fractions having a common denominator and add the numerators.

To subtract fractions, reduce the fractions to similar fractions having a common denominator and subtract the numerators.

Example

Add: $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{1}{4}$.

Solution

$$\begin{array}{c|c|r} \frac{1}{2} & & 6 \\ \frac{2}{3} & 12 & 8 \\ \frac{1}{4} & & \underline{3} \\ & & 17 \end{array}$$

$$\frac{17}{12} = 1\frac{5}{12}$$

Explanation. Inspection shows that 12 is the least common denominator. $\frac{1}{2}$ is $\frac{6}{12}$, $\frac{2}{3}$ is $\frac{8}{12}$, and $\frac{1}{4}$ is $\frac{3}{12}$. Adding the numerators of the similar fractions gives 17, and $\frac{17}{12}$ is $1\frac{5}{12}$.

Example	*Solution*
Subtract: $\frac{3}{4} - \frac{5}{16}$	$\frac{3}{4} = \frac{12}{16}$ $\frac{12}{16} - \frac{5}{16} = \frac{7}{16}$

Multiplication of fractions. (*a*) To multiply a fraction by a whole number, multiply the numerator or divide the denominator of the fraction by the whole number.

Example	*Solution*
Multiply $6 \times \frac{5}{12}$.	$6 \times \frac{5}{12} = \frac{30}{12} = 2\frac{1}{2}$ or $12 \div 6 = 2$, and $\frac{5}{2} = 2\frac{1}{2}$

(*b*) To multiply a whole number by a fraction, multiply the whole number by the numerator of the fraction and write the product over the denominator. Cancel when possible.

Example	*Solution*
Find $\frac{2}{5}$ of 35.	$\frac{2}{5} \times 35 = \frac{70}{5} = 14$ or $\dfrac{2 \times \overset{7}{\cancel{35}}}{\cancel{5} \times 1} = 14$

(*c*) To multiply a fraction by a fraction, multiply the numerators to obtain the numerator of the answer, and multiply the denominators to obtain the denominator of the answer. Cancel when possible.

Example	*Solution*
Find $\frac{2}{3}$ of $\frac{15}{16}$.	$\frac{2}{3} \times \frac{15}{16} = \frac{30}{48} = \frac{5}{8}$ or $\dfrac{\cancel{2} \times \overset{5}{\cancel{15}}}{\cancel{3} \times \underset{8}{\cancel{16}}} = \dfrac{5}{8}$

(*d*) To multiply a mixed number by a mixed number, reduce each mixed number to an improper fraction and proceed as in (*c*).

Example	*Solution*
Find the product of: $3\frac{1}{2} \times 4\frac{1}{8}$	$\frac{7}{2} \times \frac{33}{8} = \frac{231}{16} = 14\frac{7}{16}$
Find the product of: $6\frac{2}{5} \times 5\frac{1}{8}$	$\dfrac{\overset{4}{\cancel{32}} \times 41}{5 \times \cancel{8}} = \dfrac{164}{5} = 32\dfrac{4}{5}$

Problems

Find:

1. $9 \times \frac{5}{18}$	**3.** $\frac{2}{5}$ of 35	**5.** $\frac{2}{3}$ of $\frac{15}{16}$	**7.** $3\frac{1}{4} \times 4\frac{1}{2}$
2. $24 \times \frac{3}{4}$	**4.** $\frac{5}{12}$ of 16	**6.** $\frac{4}{5}$ of $\frac{25}{28}$	**8.** $12\frac{3}{4} \times 8\frac{1}{4}$

Division of fractions. (*a*) To divide a fraction by a whole number, divide the numerator or multiply the denominator by the whole number.

Example — Divide $\frac{25}{28}$ by 5.

Solution

$25 \div 5 = 5$ Answer: $\frac{5}{28}$

or

$$\frac{\overset{5}{\cancel{25}} \times 1}{28 \times \cancel{5}} = \frac{5}{28}$$

(*b*) To divide any quantity—a whole number, a mixed number, or a fraction, by a fraction, invert the divisor and multiply.

Example — Divide 8 by $\frac{2}{3}$.

Solution

$$\frac{\overset{4}{\cancel{8}} \times 3}{1 \times \cancel{2}} = 12$$

Example — Divide $16\frac{1}{4}$ by $\frac{5}{6}$.

Solution

$$\frac{\overset{13}{\cancel{65}} \times \overset{3}{\cancel{6}}}{\underset{2}{\cancel{4}} \times \cancel{5}} = \frac{39}{2} = 19\frac{1}{2}$$

Example — Divide $\frac{3}{4}$ by $\frac{1}{2}$.

Solution

$$\frac{3 \times \cancel{2}}{\underset{2}{\cancel{4}} \times 1} = \frac{3}{2} = 1\frac{1}{2}$$

Problems

Divide:

a. $\frac{18}{35}$ by 3	**c.** 8 by $\frac{2}{3}$	**e.** $16\frac{3}{4}$ by $\frac{1}{8}$	**g.** $3\frac{1}{2}$ by $1\frac{1}{2}$
b. $\frac{36}{43}$ by 9	**d.** 9 by $\frac{3}{5}$	**f.** $18\frac{4}{5}$ by $\frac{5}{9}$	**h.** $9\frac{3}{4}$ by $3\frac{1}{4}$

1. How many pieces of wire each $8\frac{3}{4}$ inches long can be cut from 40 feet of wire?

2. If $\frac{2}{3}$ of a ton of coal costs $12.75, what is the cost of one ton?

3. How many sash weights each weighing $2\frac{1}{2}$ pounds can be cast from 120 pounds of pig iron, if $\frac{1}{8}$ of the quantity of pig iron is wasted in the casting operation?

4. A room is $18\frac{5}{8}$ feet long and $14\frac{1}{2}$ feet wide. The width of the room is what part of the length of the room?

5. A carpenter has a board that is 20 feet long, but it is $\frac{1}{4}$ longer than he needs. How long a board does he need?

6. What is the cost of $7\frac{1}{2}$ tons of coal at $\$14\frac{1}{4}$ a ton?

7. A house and lot are valued at \$6,600. If the lot is worth $\frac{3}{8}$ as much as the house, what is the value of each?

8. If a man can earn \$$2\frac{3}{4}$ a day, how long will it take him to earn \$$46\frac{3}{4}$?

9. A table is 20 feet long. How many people can be seated on the two sides if you allow $1\frac{2}{3}$ feet for each person?

10. Henry's time book shows that his working time for one week was as follows: Monday, $7\frac{1}{2}$ hours; Tuesday, $8\frac{1}{4}$ hours; Wednesday, 8 hours; Thurdsay, $9\frac{1}{4}$ hours; Friday, $8\frac{1}{2}$ hours; Saturday, $6\frac{2}{3}$ hours.

He is paid straight time for 8 hours or less and time and a half for hours in excess of 8 each day other than Saturday, when he receives double-time pay for hours worked. How much did he earn at \$$\frac{5}{8}$ an hour?

11. The shipping clerk reported that he dispatched 320 packages averaging $28\frac{3}{4}$ pounds each. What was the total weight of packages dispatched?

12. A cubic foot of water weighs $62\frac{1}{2}$ pounds, and there are approximately $7\frac{1}{2}$ gallons to the cubic foot. Estimate the weight of water that a 10-gallon keg will contain.

To find the product of any two mixed numbers ending in $\frac{1}{2}$.

(a) *When the sum of the whole numbers is an even number.* To the product of the whole numbers, add one-half of their sum, and annex $\frac{1}{4}$.

Example

Multiply $24\frac{1}{2}$ by $8\frac{1}{2}$.

Solution

$$\begin{array}{rl} 24\frac{1}{2} & \\ \underline{8\frac{1}{2}} & \\ 192 & (8 \times 24) \\ \underline{16} & (\frac{1}{2} \text{ of the sum of 24 and 8}) \\ 208\frac{1}{4} & (\frac{1}{4} \text{ annexed}) \end{array}$$

Problems

Multiply:

1. $8\frac{1}{2}$ by $4\frac{1}{2}$.
2. $12\frac{1}{2}$ by $8\frac{1}{2}$.
3. $28\frac{1}{2}$ by $12\frac{1}{2}$.
4. $16\frac{1}{2}$ by $14\frac{1}{2}$.
5. $18\frac{1}{2}$ by $18\frac{1}{2}$.
6. $10\frac{1}{2}$ by $34\frac{1}{2}$.

(b) *When the sum of the whole numbers is an odd number.* To the product of the whole numbers, add one-half of their sum, less 1, and annex $\frac{3}{4}$.

Example

Multiply $15\frac{1}{2}$ by $6\frac{1}{2}$.

Solution

$$\begin{array}{rl} 15\frac{1}{2} & \\ \underline{6\frac{1}{2}} & \\ 90 & (6 \times 15) \\ \underline{10} & (\frac{1}{2} \text{ of } 15 + 6 - 1) \\ 100\frac{3}{4} & (\frac{3}{4} \text{ annexed}) \end{array}$$

Problems

Multiply:

1. $18\frac{1}{2}$ by $5\frac{1}{2}$.	**3.** $38\frac{1}{2}$ by $5\frac{1}{2}$.	**5.** $23\frac{1}{2}$ by $4\frac{1}{2}$.
2. $14\frac{1}{2}$ by $7\frac{1}{2}$.	**4.** $13\frac{1}{2}$ by $8\frac{1}{2}$.	**6.** $19\frac{1}{2}$ by $6\frac{1}{2}$.

To multiply a mixed number by a mixed number.

Example

Multiply $524\frac{1}{2}$ by $27\frac{1}{3}$.

Solution

$$
\begin{array}{rll}
524\frac{1}{2} & & \\
\underline{27\frac{1}{3}} & & \\
14148 & \underline{6} & = \text{common denominator of fractions} \\
174\frac{2}{3} & 4 & \left.\right\} \\
13\frac{1}{2} & 3 & \left.\right\} = \text{numerators of changed fractions} \\
\underline{\frac{1}{6}} & \underline{1} & \left.\right\} \\
14336\frac{1}{3} & \frac{8}{6} & = 1\frac{1}{3}
\end{array}
$$

Explanation. Multiply 524 by 27, obtaining the first part of the answer, 14,148. Next, take $\frac{1}{3}$ of 524, obtaining $174\frac{2}{3}$. Then take $\frac{1}{2}$ of 27, obtaining $13\frac{1}{2}$. Finally, take $\frac{1}{3}$ of $\frac{1}{2}$, obtaining $\frac{1}{6}$. Add the four partial products, and the complete product is $14,336\frac{1}{3}$.

Problems

Multiply:

1. $247\frac{2}{3}$ by $39\frac{1}{4}$.	**3.** $59\frac{1}{7}$ by $15\frac{1}{3}$.	**5.** $181\frac{3}{7}$ by $6\frac{2}{3}$.
2. $849\frac{1}{6}$ by $28\frac{1}{2}$.	**4.** $176\frac{5}{8}$ by $34\frac{2}{7}$.	**6.** $56\frac{1}{2}$ by $12\frac{2}{3}$.

Decimal fractions. A decimal fraction is a fraction whose denominator is some power of ten, indicated by a decimal point placed just to the right of the units' place. Thus, .1 is $\frac{1}{10}$, .05 is $\frac{5}{100}$, and .25 is $\frac{25}{100}$ or $\frac{1}{4}$.

Addition and subtraction. To add or to subtract decimals, write the numbers so that the decimal points fall vertically and proceed as in whole numbers.

Example

Add: .01, 4.72, 78.25, and .005.

Solution

$$
\begin{array}{r}
.01 \\
4.72 \\
78\ 25 \\
\underline{.005} \\
82.985
\end{array}
$$

Example

Subtract:
47.02 − .92

Solution

$$
\begin{array}{r}
47.02 \\
\underline{92} \\
46.10
\end{array}
$$

Problems

1. Add: 25.679, .0356, 2.78, and .017.

2. Add: 136.2, 28.348, .004, and 1.356.

3. Subtract: 13.48 from 27.049.

4. Subtract: .003 from .47.

Multiplication. To multiply decimal fractions, multiply as in whole numbers and point off as many decimal places in the product as there are places in both multiplicand and multiplier.

Example	*Solution*
Multiply 3.06 × .8.	$\begin{array}{r} 3\ 06 \\ .8 \\ \hline 2.448 \end{array}$

Explanation. Since there are 3 decimal places in both the multiplicand and the multiplier, point off three decimal places in the product.

Example	*Solution*
Multiply: 23.8564 by 6.72	$\begin{array}{r} 23.8564 \\ 6\ 72 \\ \hline 477128 \\ 1669948 \\ 1431384 \\ \hline 160315008 \end{array}$

Explanation. As there are 6 decimal places in the multiplicand and the multiplier, point off six decimal places in the product. The answer is 160.315008. Rough check: 24 × 7 = 168.

Division. Proceed as with whole numbers, annexing zeros to the dividend if necessary. The number of decimal places in the quotient must equal the number in the dividend minus the number in the divisor.

Example	*Solution*
Divide: 54.864 by .24	$\begin{array}{l} .24)54.864(228.6 \\ \quad\;\; 6\ 8 \\ \quad\;\; 2\ 06 \\ \quad\quad\;\; 144 \\ \quad\quad\quad\; 0 \end{array}$

Explanation. Divide by writing the remainders only. The quotient is 2286. As there are three decimal places in the dividend and two decimal places in the divisor, point off one decimal place in the quotient. The answer is, therefore, 228.6.

Example	*Solution*
Divide: 256.7894 by 5.23	$\begin{array}{r} 49.099 \\ 5.23\overline{)256.78940} \\ 47\ 58 \\ 5194 \\ 4870 \\ 163 \end{array}$

Explanation. Predetermine the placing of the decimal. As there are two decimals in the divisor, place the decimal point over the third decimal place in the dividend. Place the first figure of the quotient over the last figure of the partial dividend. One zero has been annexed to the dividend in order to obtain a quotient to three decimals. Rough check: 49 × 5 = 245.

Problems

Multiply:	Divide:
1. 34.278 × 1.45	**6.** 5.8769 by 1.34
2. 395.264 × .035	**7.** .0084 by 1.5
3. 74.26 by .00423	**8.** 45.87 by .0056
4. .056 by .083	**9.** 8.45 by 25.3
5. 18.42 × .045	**10.** 956 by 4.87

To abbreviate decimal multiplication when a given number of decimal places is required. It is a waste of time to carry out decimal multiplication to a denomination smaller than that in which the data are expressed; often it is unnecessary to carry it beyond the third or fourth decimal.

Example

Multiply 4.7892 by 3.1765, and obtain the answer correct to four decimal places.

Solution

4 7892		= multiplicand
5 6713		= multiplier reversed
14.3676		= 4.7892 × 3.
.4789	2	= 4.789~~2~~ × .1
.3352	4 ~~4~~	= 4.78~~92~~ × 07
. 287	3 ~~5~~ 2	= 4 7~~892~~ × 006
23	9 ~~1~~ ~~6~~ ~~0~~	= 4.7~~892~~ × 0005
15.2128	~~9~~ ~~3~~ ~~8~~ ~~0~~	

Explanation. The multiplier, 3.1765, is written in the reverse order, 56713, the units' digit being placed under the lowest order of the multiplicand that is desired in the product—ten thousandths. Multiply by each digit of the reversed multiplier, beginning with that digit of the multiplicand which stands directly above the digit of the multiplier used, taking care to include the digit carried over from the multiplication of the one (or two) rejected digits at the right.

Example

Multiply 4.7869347 by 7.25, and obtain the product correct to three decimal places.

Solution

4 786 9347	
527	
33 508 3	= 4 7869 × 7
.957 2	= 4 786~~9~~ × 2
239 0	= 4.78~~69~~ × 5
34 704 5	

Problems

Multiply:

1. 5.987654 by 3.147, obtaining the product correct to the 4th decimal.
2. $3.596\frac{2}{3}$ by 14.57, obtaining the product correct to the 3rd decimal.
3. $184.28\frac{4}{7}$ by 3.145, obtaining the product correct to the 4th decimal.
4. 44.187542 by 6.2434, obtaining the product correct to the 3rd decimal.

Division of decimals. Division of decimals may often be abbreviated, especially when the divisor is given to a greater number of decimal places than are contained in the dividend, and when only three or four decimal places are essential in the quotient.

Example

Divide 4.39876 by 2.4871934, and obtain the quotient correct to three decimal places.

Solution

Ordinary Method	*Abbreviated Method*
2.4871934)4.398 7600 (1.768	2.487 ~~1934~~)4.398 7(1.768
2 487 1934	2 487 2
1 911 56660	1 911 5
1 741 03538	1 741 0
170 531220	170 5
149 231604	149 2
21 2996160	21 3
19 8975472	19 9
1 4020688	1 4

Explanation. Observation of the ordinary method shows that the third decimal place in the quotient is not affected by the digit in the third decimal place in the divisor (except through the digits carried).

Since the units' digit of the divisor is contained in the units' digit of the dividend, the first digit in the quotient is in the units' place, and as three decimal places are required, the quotient will contain four digits. Therefore, the last four digits of the divisor will not affect the quotient, except through the digits carried over.

The first four digits of the divisor, 2.487, are contained once in 4.398. Multiplication of that part of the divisor used, by the quotient digit (including the digit carried over from the one or two following digits—in this case considering the 9 as a unit and adding it to the 1, making 2) gives 2487 2, and this result deducted from the previous dividend leaves 1911 5 for the new dividend.

Cancel the right-hand digit, 7, of the divisor, and divide 1911 by 248, obtaining the quotient 7. Multiplying the divisor by 7 (and including the carrying digit) gives 1741 0, and subtracting leaves a new dividend of 170 5.

Cancel another digit, 8, of the divisor, and divide by 24. This is contained 6 times in 170. The product (including the digit carried over) is 149 2, and this product subtracted leaves a new dividend of 21 3.

Cancel another digit, 4, of the divisor. Divide 21 by 2, using the carried digit; the result is 8. The new product is 19 9, and this product subtracted from 21 3 leaves a remainder of 1 4.

Example

Divide 8.47 by 31.76983476, and obtain the quotient correct to three decimal places.

Solution

```
31.769 83476)8 4700(.266
             6 3540  =  31.769  (8) × .2, or 6.3539(6).   Use 6.3540.
             ------
             2 1160
             1 9062  =  31.76  (98) × .06, or 1.9061(8).  Use 1.9062.
             ------
               2098
               1906  =  31.7   (69) × .006, or .1906(1).  Use .1906.
               ----
                192
```

Problems

Divide:

1. 4.3954 by 37.265872, obtaining the quotient correct to the 3rd decimal.
2. 65.157 by 4.4976348, obtaining the quotient correct to the 4th decimal.
3. 1.297648 by 15.782643, obtaining the quotient correct to the 3rd decimal.
4. 3.489765 by .28765431, obtaining the quotient correct to the 3rd decimal

To change a decimal fraction to an equivalent common fraction. Write the denominator of the decimal, omit the decimal point, and reduce to lowest terms. Thus, to reduce to common fractions in lowest terms or to mixed numbers:

$$.75 = \tfrac{75}{100} = \tfrac{3}{4} \qquad .025 = \tfrac{25}{1000} = \tfrac{1}{40}$$
$$6.25 = 6\tfrac{25}{100} = 6\tfrac{1}{4} \qquad 4.125 = 4\tfrac{125}{1000} = 4\tfrac{1}{8}$$

To change a common fraction to a decimal. A common fraction may be regarded as an indicated division. Thus: $\frac{2}{5}$ may be regarded as $2 \div 5$; therefore, $\frac{2}{5}$ expressed as a decimal is .4; similarly, $\frac{1}{7}$ is $.14\frac{2}{7}$, $\frac{3}{8}$ is .375, and $\frac{7}{16}$ is .4375.

Aliquot parts. An aliquot part of any number is a number that is contained in it an integral number of times. Thus, 5, 10, 20, and 50 are aliquot parts of 100; that is, $5 = \frac{1}{20}$ of 100, $10 = \frac{1}{10}$ of 100, and so forth.

The use of aliquot parts. As a means of saving time in multiplication and in division, it is useful to know the decimal equivalents of common fractions, or, conversely, to know the common fraction equivalents of decimal fractions. Aliquot parts are of value in addition and subtraction if an adding machine or a calculating machine is used, because machines are not adapted for general work involving common fractions.

TABLE OF ALIQUOT PARTS OF 1

Common Fraction	*Decimal Equivalent*	*Common Fraction*	*Decimal Equivalent*
$\frac{1}{2}$	$.50$	$\frac{1}{9}$	$.11\frac{1}{9}$
$\frac{1}{3}$	$.33\frac{1}{3}$	$\frac{1}{10}$	$.10$
$\frac{2}{3}$	$.66\frac{2}{3}$	$\frac{1}{11}$	$.09\frac{1}{11}$
$\frac{1}{4}$	$.25$	$\frac{1}{12}$	$.08\frac{1}{3}$
$\frac{3}{4}$	$.75$	$\frac{5}{12}$	$.41\frac{2}{3}$
$\frac{1}{5}$	$.20$	$\frac{7}{12}$	$.58\frac{1}{3}$
$\frac{1}{6}$	$.16\frac{2}{3}$	$\frac{11}{12}$	$.91\frac{2}{3}$
$\frac{5}{6}$	$.83\frac{1}{3}$	$\frac{1}{15}$	$.06\frac{2}{3}$
$\frac{1}{7}$	$.14\frac{2}{7}$	$\frac{1}{16}$	$.06\frac{1}{4}$
$\frac{2}{7}$	$28\frac{4}{7}$	$\frac{3}{16}$	$.18\frac{3}{4}$
$\frac{3}{7}$	$.42\frac{6}{7}$	$\frac{5}{16}$	$.31\frac{1}{4}$
$\frac{4}{7}$	$.57\frac{1}{7}$	$\frac{7}{16}$	$.43\frac{3}{4}$
$\frac{5}{7}$	$71\frac{3}{7}$	$\frac{9}{16}$	$.56\frac{1}{4}$
$\frac{6}{7}$	$.85\frac{5}{7}$	$\frac{11}{16}$	$.68\frac{3}{4}$
$\frac{1}{8}$	$.12\frac{1}{2}$	$\frac{15}{16}$	$.93\frac{3}{4}$
$\frac{3}{8}$	$37\frac{1}{2}$	$\frac{1}{25}$	$.04$
$\frac{5}{8}$	$.62\frac{1}{2}$	$\frac{1}{32}$	$.03\frac{1}{8}$
$\frac{7}{8}$	$87\frac{1}{2}$	$\frac{3}{32}$	$.09\frac{3}{8}$

The fractions in the above table can be extended as decimals as far as the work demands.

Problems

Express the following as decimal fractions; non-terminating fractions should be carried to the sixth decimal place and the common fraction annexed:

$\frac{2}{3}$	$\frac{1}{9}$	$\frac{1}{32}$	$\frac{3}{14}$	$\frac{5}{16}$	$\frac{11}{12}$
$\frac{3}{4}$	$\frac{1}{3}$	$\frac{1}{8}$	$\frac{2}{7}$	$\frac{6}{7}$	$\frac{2}{9}$
$\frac{5}{6}$	$\frac{1}{15}$	$\frac{1}{7}$	$\frac{15}{16}$	$\frac{1}{14}$	$\frac{1}{10}$
$\frac{8}{9}$	$\frac{1}{16}$	$\frac{2}{5}$	$\frac{1}{21}$	$\frac{3}{7}$	$\frac{1}{20}$
$\frac{5}{7}$	$\frac{4}{7}$	$\frac{3}{16}$	$\frac{5}{8}$	$\frac{5}{6}$	$\frac{3}{32}$

Multiplication by aliquot parts.

Example

Find $16\frac{2}{3}\%$ of \$475.34.

Solution

$$\begin{array}{r} 6\overline{)\$475.34} \\ \$79.22 \end{array}$$

Explanation. Since $.16\frac{2}{3}$ equals $\frac{1}{6}$, find $\frac{1}{6}$ of \$475.34.

Example

Find the cost of 256 units at $37\frac{1}{2}$¢ each.

Solution

$$256 \times \tfrac{3}{8} \times \$1 = \$96$$

Explanation. $37\frac{1}{2}$¢ is $\frac{3}{8}$ of \$1. Therefore, $256 \times \frac{3}{8} \times \$1 = \$96$.

Problems

Extend the following items mentally:

1. 72 @ .12$\frac{1}{2}$	**9.** 18 @ .33$\frac{1}{3}$	**17.** 64 @ .25	**25.** 72 @ .83$\frac{1}{3}$
2. 45 @ .11$\frac{1}{9}$	**10.** 39 @ .66$\frac{2}{3}$	**18.** 27 @ .22$\frac{2}{9}$	**26.** 32 @ .87$\frac{1}{2}$
3. 24 @ .08$\frac{1}{3}$	**11.** 55 @ .09$\frac{1}{11}$	**19.** 32 @ .18$\frac{3}{4}$	**27.** 36 @ .41$\frac{2}{3}$
4. 36 @ .50	**12.** 16 @ .75	**20.** 96 @ .03$\frac{1}{8}$	**28.** 27 @ .44$\frac{4}{9}$
5. 15 @ .06$\frac{2}{3}$	**13.** 49 @ .28$\frac{4}{7}$	**21.** 48 @ .56$\frac{1}{4}$	**29.** 12 @ .75
6. 75 @ .93$\frac{1}{3}$	**14.** 32 @ .43$\frac{3}{4}$	**22.** 60 @ .58$\frac{1}{3}$	**30.** 14 @ .07$\frac{1}{7}$
7. 48 @ .16$\frac{2}{3}$	**15.** 28 @ .57$\frac{1}{7}$	**23.** 48 @ .37$\frac{1}{2}$	**31.** 18 @ .16$\frac{2}{3}$
8. 32 @ .06$\frac{1}{4}$	**16.** 24 @ .62$\frac{1}{2}$	**24.** 35 @ .14$\frac{2}{7}$	**32.** 16 @ .87$\frac{1}{2}$

Division by aliquot parts. It is difficult to divide a number by a mixed number. If the divisor is an aliquot part, the quotient may be found by multiplication, as follows:

Example

Divide 4,875 by 16$\frac{2}{3}$.

Explanation. Since 16$\frac{2}{3}$ is $\frac{1}{6}$ of 100, divide 4,875 by $\frac{1}{6}$ of 100, or $\frac{100}{6}$. This is the same as multiplying by $\frac{6}{100}$. Therefore, divide by 100 by pointing off two decimal places from the right, and multiply the result by 6. The answer is 292.50, or 292$\frac{1}{2}$.

Solution
$$\begin{array}{r} 48\ 75 \\ 6 \\ \hline 292\ 50 \end{array}$$

Example

The production cost of 1,250 units is \$3,170. Find the cost per unit.

Explanation. 1,250 is $\frac{1}{8}$ of 10,000. Divide \$3,170 by 10,000 by pointing off 4 decimal places from the right; then multiply the result by 8. The cost per unit is found to be \$2.536.

Solution
$$\begin{array}{r} 3170 \\ 8 \\ \hline 2\ 5360 \end{array}$$

Problems

Divide:

1. 1,342 by 11$\frac{1}{9}$.	**3.** 3,126 by 33$\frac{1}{3}$.	**5.** 158 by 6$\frac{1}{4}$.
2. 2,578 by 12$\frac{1}{2}$.	**4.** 384 by 25.	**6.** 4,275 by 14$\frac{2}{7}$.

Problems

1. A manufacturer pays dividends amounting to $\frac{3}{16}$ of his capital. If the dividends amount to \$37,500, what is the capital?

2. A fuel dealer had 36 cords of wood and sold $\frac{2}{9}$ of it. How many cords did he sell?

3. If a merchant buys an article for \$12$\frac{1}{2}$ and sells it for \$16, the profit is what fraction of the selling price? What fraction of the cost price?

4. A crate containing 10 dozen oranges cost \$4.50. If they are sold at the rate of 65 cents a dozen, but $\frac{1}{2}$ dozen are spoiled, the profit is what fraction of the selling price?

5. A man has \$37$\frac{1}{2}$ and spends \$12$\frac{1}{2}$. What fraction of his money does he keep?

6. A factory normally employed 48 men. During a dull period 16 received temporary lay-offs. What fraction of the force continued to work?

7. The last reading of a gas meter was 67,324 cu. ft.; the previous reading was 64,815 cu. ft. At $1.45 a thousand cubic feet, find the amount of the gas bill.

8. An investment of $18,000 produces an annual income of $720. At the same rate, what should an investment of $25,000 produce?

9. Tires costing $18.75 were installed when the speedometer registered 18,985 miles. The four tires were replaced when the speedometer registered 34,652 miles. $1.00 was allowed for each old tire. What was the average tire cost per mile, correct to the nearest tenth of a mill?

10. An excavation 8 feet in depth required the removal of 5,328 cu. ft. of earth and rock. The average depth of earth was 5 ft., and the cost of earth removal was $$1\frac{1}{4}$ a cu. yd. The remainder was rock and cost $$4\frac{7}{8}$ a cu. yd. for removal. What was the cost of making the excavation?

CHAPTER 5

Percentage

Relation between percentage and common and decimal fractions. Percentage is a continuation of the subject of fractions. It is the process of computing by hundredths, but instead of the term *hundredths*, the Latin expression *per cent* is used. The sign (%) generally replaces the words *per cent*, thus, 5%, 10%, and so forth.

Any per cent may be expressed either as a common fraction or as a decimal, thus:

	Common Fraction	*Decimally*
1%	$\frac{1}{100}$	.01
5%	$\frac{5}{100}$	.05
$12\frac{1}{2}$%	$\frac{12\frac{1}{2}}{100}$ or $\frac{125}{1000}$	$.12\frac{1}{2}$ or .125
100%	$\frac{100}{100}$	1
300%	$\frac{300}{100}$	3.
$\frac{1}{2}$%	$\frac{\frac{1}{2}}{100}$ or $\frac{5}{1000}$	$.00\frac{1}{2}$ or .005
.05%	$\frac{\frac{5}{100}}{100}$ or $\frac{5}{10,000}$	.0005

Care should be taken in writing per cents. Do not write both the sign and the decimal point; thus, 2% and .02 are the same, but 2% and .02% are widely different, since the first is equivalent to $\frac{1}{50}$ and the second to $\frac{1}{5000}$.

Applications. Percentage admits of applications in many fields. Business operations are guided by carefully prepared statistics, and the relationships of items in statistics are often more clearly reflected when they are expressed in terms of percentage. There are numerous problems involving percentage besides those having to do with financial considerations, such as finding the per cent of increase or decrease in volume; per cent of shrinkage of material; per cent of waste in manufacturing operations; per cent of yield of crops.

Definitions. The *base* is the number or quantity represented by 100%. The base may be, for example, total sales, total

expenses, the face value of a note, the par value of a bond, pounds of material used, capacity, and so forth.

The *rate* is the number of hundredths, or the per cent. The rate may be, for example, 6% or 25%, which are written decimally as .06 and .25.

The *percentage* is the product of the base and the rate. The percentage may be, for example, the interest cost of a sum of money, the departmental portions of an expense item, the increase in pounds of material used, and the like.

Fundamental processes. In percentage and its application, three fundamental mathematical principles are involved, namely: (1) to find a given per cent of a number; (2) to find what per cent one number is of another; and (3) to find a number when a certain per cent of it is known.

Computations. Computations in percentage are based on these principles.

Principle 1. The percentage is the product of the base and the rate.

Base × Rate = Percentage

Example

6% interest on $500 is $30. (500 × .06 = 30)

Problems

In the following, convert the per cent either to a common fraction or to a decimal fraction, whichever is the easier.

Find:

1. 25% of 5,280 ft.	**6.** $2\frac{2}{5}$% of 180 lbs.
2. 10% of 846 lbs.	**7.** $\frac{3}{8}$% of 240 gal.
3. $16\frac{2}{3}$% of 24 bu.	**8.** $\frac{3}{4}$% of $5,000.
4. $37\frac{1}{2}$% of $60.	**9.** 20% of 95 yds.
5. 80% of 120 pp.	**10.** $14\frac{2}{7}$% of 42 in.

11. If an expense item of $16.00 is reduced $6\frac{1}{4}$%, what will be the amount of this item after the reduction?

12. A commission of $12\frac{1}{2}$% was earned on a $240 sale. What was the commission?

13. A sample of grain showed $2\frac{2}{3}$% weed seed. How many bushels of weed seed are in 600 bushels of this grain?

14. An item sells for 40 cents. What will be the selling price after a reduction of 15%?

15. Anticipated requirements for copper will exceed the manufacturer's stock by 35%. If 185 pounds are on hand, how many pounds will have to be purchased?

Principle 2. The rate may be found by dividing the percentage by the base.

Percentage ÷ Base = Rate.

Example

$\$30 \div \$500 = .06$ or 6%.

Problems

In the following find what per cent of:

1. 72 is 24
2. 60 is 50
3. 180 is 120
4. 360 is 90
5. 50 is 20
6. 12 is 20
7. 64 is 8
8. 90 is 10
9. 150 is 25
10. 125 is 25

11. Last year's taxes on a house were \$520. This year's taxes were \$640 What per cent were this year's taxes of last year's taxes?

12. A pile of lumber contained 4,500 feet, and 3,300 feet were used. What per cent remained?

13. Wages are increased from \$1.50 an hour to \$1.75 an hour. Find the per cent of increase.

14. A new style of packaging reduced the shipping weight from 130 lbs. to 121 lbs. What was the per cent of saving in shipping weight?

15. The inspector rejected 5 items out of 140 produced. What was the per cent of rejects?

Principle 3. The base may be found by dividing the percentage by the rate.

Percentage ÷ Rate = Base.

Example

$30 \div .06 = 500.$

Problems

Find the number of which:

1. 25 is 20%
2. 125 is $16\frac{2}{3}\%$
3. 240 is 75%
4. 48 is $\frac{1}{4}\%$
5. 72 is $12\frac{1}{2}\%$
6. 86 is 43%
7. 374 is 17%
8. 375 is $\frac{5}{8}\%$
9. $4\frac{1}{2}$ is $\frac{3}{8}\%$
10. 26 is 40%

11. The fire insurance premium on a house was \$22.50. The house was insured for 80% of its value at $\frac{3}{8}\%$. Find the value of the house.

12. Sales increased each year over the preceding year as follows: 15% the second year, 20% the third year, and 25% the fourth year. If the fourth year's sales were \$21,562.50, what were the first year's sales?

13. A bankrupt can pay his creditors 72 cents on the dollar. If his assets are \$13,475.28, what are his liabilities?

14. The gross income of a rental property is \$1,800 a year. Expenses are \$500. If the net income is a return of $6\frac{1}{2}\%$ on the investment, find the value of the property.

15. One workman completes a unit in $7\frac{1}{2}$ hours. Another workman completes a similar unit in $5\frac{3}{4}$ hours. The first workman took what per cent more time than the second workman to complete the unit?

Miscellaneous Problems

1. A machine that cost $50 was marked up 30%. What was the marked price?

2. After a clerk's salary was increased $6\frac{1}{4}$%, he received $850 a year. What was his former salary?

3. A 4-apartment building cost $18,000. Repairs average $1\frac{1}{3}$% of the cost; taxes, $2\frac{1}{4}$%; insurance on 90% valuation, $\frac{3}{8}$%; other expenses amount to $114.25. What should the annual rental income be in order to return the owner 8% on his investment? What should be the average monthly rental of each apartment?

4. A product shrinks 16% in processing. How many pounds of raw material will be required to process 252 pounds of finished product?

5. A creditor received $637.73 from a bankrupt estate paying 68 cents on the dollar. What was the creditor's loss on the account?

Daily record of departmental sales. The following tabulation is designed to show the total daily sales by departments, and the total sales for the week, both by departments and for the business as a whole. After Saturday's sales have been entered, the total departmental sales for the week may be found and also the per cent that each department's sales is of total sales. The per cent that each day's sales is of total sales for the week is also obtainable.

DAILY RECORD OF DEPARTMENTAL SALES

Dept.	*Mon.*	*Tues.*	*Wed.*	*Thurs.*	*Fri.*	*Sat.*	*Total*	*Per Cent*
A	$475.86	$275 83	$329 86	$424 83	$387 92	$412 15	....	......
B	324.18	174 82	274 19	285 27	304.14	319 28	......	......
C	456.19	259 80	179 86	258 24	286 39	305 74	......	......
D	421 40	268 75	142.56	280 22	178 90	260 57	......	......
E	175 60	125 34	156 85	210 05	162 50	187 50	......	......
Total	......	......	......	......	......	......		100 00%
Per Cent	......	......	......	......	......	......	100 00%	

Problem

Prepare a form similar to the above, enter the sales in the proper columns, and find: (*a*) the total sales for each day in all departments (add downward); (*b*) the total sales for each department for the week (add across); (*c*) in two ways, the total sales in all departments for the entire week; (*d*) the per cent of grand total sales made each day (total for each day divided by the grand total); (*e*) the per cent of grand total sales made in each department (total of each department divided by the grand total).

Per cent of returned sales by departments. In some lines of business it is important to keep a close check on the volume of

returned sales. This may be done advantageously by means of per cents derived from tabulated results.

Problem

Prepare a form similar to the following, enter the data, and find: (*a*) the net sales for each department, and the net sales for all the departments; (*b*) the per cent of returned sales in each department, and the total per cent of returned sales.

SALES AND RETURNED SALES BY DEPARTMENTS

Dept.	*Sales*	*Returned Sales*	*Net Sales*	*Per Cent of Sales Returned*
A	$ 24,863 95	$ 756 82		
B	110,356 80	1,328 95		
C	53,768 21	975 32		
D	16,135 40	628 74		
E	9,356 24	256 48		
Total				

Clerk's per cent of average sales. As a measure of efficiency, the following tabulation may be made for a department, and each clerk's weekly or monthly sales compared with the average weekly or monthly sales.

MONTHLY SALES OF CLERKS—DEPT. *A*

Clerk's Number	*Monthly Sales*	*Per Cent of Average*
1	$2,756 80	
2	1,954 36	
3	2,075 83	
4	2,634 87	
5	2,315 62	
Total		100 00%
Average		

Problem

Prepare a form similar to the above, enter the data, and find: (*a*) the total monthly sales; (*b*) the average monthly sales per clerk; and (*c*) what per cent each clerk's sales are of the average sales per clerk.

Per cent of income by source. In accounting for the income of a public service enterprise, it is desirable to show the per cent of income from each source when the company's activities are varied.

Problems

1. In the following tabulation of gross earnings of a public utility corporation, find what per cent the earnings from each source are of the total gross earnings.

Source	*Gross Earnings*	*Per Cent*
Electric light and power	$15,817,324 00	
Electric and steam railroads	6,763,656 00	
City railways and bus lines	4,248,824 00	
Gas	3,191,720 00	
Heat	672,394 00	
Bridges	589,691 00	
Ice	254,670 00	
Water	88,303 00	
Miscellaneous	21,816 00	
	$31,648,398 00	100 00%

2. In the following tabulation of the revenue from transportation of an interurban railway, find what per cent each item of revenue is of the total revenue.

REVENUE FROM TRANSPORTATION

Source	*Amount*	*Per Cent*
Passengers	$657,855 00	
Baggage	550 00	
Parlor and chair cars	9,894.00	
Special cars	25 00	
Mail	1,500 00	
Express	21,962 00	
Milk	1,666 00	
Freight	264,214 00	
Miscellaneous	269 00	
	$957,935 00	100 00%

Per cent of expense. Items of operating expenses and their relation to total expenses are more easily compared if expressed in terms of percentage.

Problems

1. In the following report of an interurban railway company, find what per cent each group of expenses is of total operating expenses.

OPERATING EXPENSES

Item	*Amount*	*Per Cent*
Way and structures	$228,690 00	
Equipment	98,979 00	
Power	105,890 00	
Conducting transportation	249,427 00	
Traffic	52,823 00	
General and miscellaneous	141,560 00	
Transportation for investment (credit)	8,403 00	
	$868,966 00	100 00%

2. In the following statement of the operating expenses of a restaurant for a period of one month, find what per cent each item of expense is of total operating expenses.

OPERATING EXPENSES

Item	*Amount*	*Per Cent*
Superintendent's labor	$ 75 00	
General labor	1,776 00	
Extra labor	160 00	
Supplies	200 00	
Electricity	58 00	
Fuel	75 00	
Laundry	103 00	
Ice	22 00	
Repairs and renewals—equipment	110 00	
Meals to employees	340 00	
Music	75 00	
Miscellaneous	66 00	
Total	$3,060 00	100 00%

Per cent of increase or decrease. Percentage is often employed to find the relation between numbers; that is, to find how much larger or smaller one number is than another.

Problems

1. In the following departmental sales tabulation, find: (*a*) the increase or the decrease in monthly sales by departments; (*b*) each department's per cent of increase or decrease (divide increase or decrease in each department by that department's monthly sales for *This Month Last Year*).

Dept.	*This Month This Year*	*This Month Last Year*	*Increase*	*Decrease*	*Per Cent Increase*	*Per Cent Decrease*
A	$2,973 69	$2,795.84				
B	1,426 85	1,852.18				
C	3,752 89	3,565 62				
D	2,581 28	2,678 15				
E	2,076 82	1,825 38				
Total						

2. In the following condensed balance sheet of a municipal railway, find the increase or decrease for each item, and also the per cent of increase or decrease.

Assets	*This Year*	*Last Year*	*Increase, Decrease*†	*Per Cent Inc., Dec.*†
Capital Assets	$ 7,912,526	$7,610,139		
Current Assets	2,174,925	2,241,395		
Deferred Assets	132,124	132,125		
Total Assets	$10,219,575	$9,983,659		
Liabilities, Reserves, and Surplus				
Funded Debt	$ 3,992,000	$4,192,000		
Current Liabilities	269,720	343,126		
Reserves	1,568,469	1,615,743		
Surplus	4,389,386	3,832,790		
Total Liabilities, etc	$10,219,575	$9,983,659		

3. In the following tabulation of advertising expenditures and direct sales resulting therefrom, compute the totals, the increase, and the per cent of increase.

	This Year		*Last Year*	
	Advertising	*Sales*	*Advertising*	*Sales*
Jan.	$2,238 00	$4,251.44	$ 1,769 64	$ 3,762.00
Feb.	2,154.00	7,461 60	1,787.96	5,067 16
Mar.	2,435 86	8,773 84	1,769 53	5,232 48
Apr.	2,425 46	7,292.12	1,840.26	7,818 00
May	2,293.12	8,709.04	1,831 70	4,867 20
June	2,035 76	8,412 28	1,825.49	4,673 12
July	none	7,383.46	none	5,083 20
Aug	none	7,656 80	none	4,454 56
Sept	none	8,227.84	none	4,650 88
Oct.	2,212 56	4,298 70	1,142 04	4,976 40
Nov	785 24	5,260 84	1,306 26	2,682 00
Dec.	none	5,683 96	none	3,542 80
Total				
Year ago			$10,607.52	$37,650.77
Increase				
% Increase				

4. In the following statement, find the increase or decrease of revenues and expenses and the per cent of increase or decrease:

	This Year	*Last Year*	*Increase, Decrease†*	*Per Cent*
Railway operating revenue	$866,197	$970,060		
Other operating revenue	8,218	7,820		
Total operating revenue	$874,415	$977,880		
Railway operating expense:				
Way and structures	$ 91,380	$ 85,569		
Equipment	64,249	61,866		
Power	108,313	114,906		
Conducting transportation	196,259	211,144		
Traffic	9,496	10,157		
General and miscellaneous	128,849	128,887		
Depreciation	17,324	44,645		
Taxes (except income taxes)	26,185	29,840		
Total	$642,055	$687,014		
Operating income	$232,360	$290,866		
Non-operating income:				
Interest funded securities	2,579	5,105		
Interest unfunded securities	7,765	6,328		
Total	$ 10,344	$ 11,433		
Gross income	$242,704	$302,299		
Deductions from gross income:				
Interest	$160,318	$161,402		
Miscellaneous	3,216	3,257		
Total	$163,534	$164,659		
Net income	$ 79,170	$137,640		

Operating statistics. The operations of a public utility engaged in transportation afford an excellent opportunity for the presentation of statistics for managerial control. The following problem has been derived from the report of such an enterprise.

Problem

From the following data, ascertain the required answers.

SECTION OF INCOME STATEMENT

Income	*This Year*	*Last Year*
Operating revenue:		
Railway operating revenue	$ 22,413,689	$ 21,678,906
Coach operating revenue	818,328	51,282
Total operating revenue	$ 23,232,017	$ 21,730,188
Non-operating income	184,273	141,767
Total revenue from all sources	$ 23,416,290	$ 21,871,955
Operating expenses:		
Railway operating expenses	$ 16,572,497	$ 15,383,494
Coach operating expenses	786,558	41,701
Total operating expenses	$ 17,359,055	$ 15,425,195
Net revenue from all sources	$ 6,057,235	$ 6,446,760
Statistics		
Railway revenue car-miles	52,863,111	48,248,330
Coach revenue coach-miles	3,529,795	157,540
Railway revenue car-hours	5,692,190	5,267,176
Railway revenue passengers	357,926,168	346,116,298
Railway transfer passengers	123,310,526	111,445,912
Railway total passengers	481,236,694	457,562,210
Coach revenue passengers	10,564,723	978,782
Coach transfer passengers	387,228	....
Coach total passengers	10,951,951	978,782
Total revenue and transfer passengers	492,188,645	458,540,992
Railway operating revenue per car-mile (cents)	....	....
Coach operating revenue per coach-mile (cents)	....	....
Railway operating expenses per car-mile (cents)	....	....
Coach operating expenses per coach-mile (cents)	....	....
Railway operating revenue per car-hour ($ and cents)	....	....
Railway operating expenses per car-hour ($ and cents)	....	....
Ratio of transfer passengers to revenue passengers—railway (per cent)	....	....
Ratio of transfer passengers to revenue passengers—coach (per cent)	....	....
Railway revenue passengers per car-mile operated	....	....
Railway transfer passengers per car-mile operated	....	....
Total railway passengers per car-mile operated	....	....
Coach revenue passengers per coach-mile operated	....	....
Coach transfer passengers per coach-mile operated	....	....

Statistics (Continued)	*This Year*	*Last Year*
Total coach passengers per coach-mile operated...	....	....
Ratio of railway operating expenses to railway operating revenue (per cent)......................	....	....
Ratio of coach operating expenses to coach operating revenue (per cent)........................	....	....

Budgeting. Percentage is also applied in budgeting, as shown by the following example from hotel accounting.

Example

Among the several items of the budget is, China and Glassware, \$3,500, to be distributed to four departments on the basis of the previous year's expense for this item in the four departments, as follows:

Department	*Per Cent*
Rooms	11 29
Restaurant	55 29
Coffee Shop	14 86
Beverages	18 56
Total	100 00%

Solution

Department	*Per Cent*	*Budget*
Rooms	11 29	\$ 395 00
Restaurant	55 29	1,935.00
Coffee Shop	14 86	520 00
Beverages	18 56	650 00
Total	100 00%	\$3,500 00

Problems

1. The following year it was found that the actual disbursements for China and Glassware amounted to \$2,280.74, and other facts were as given in the tabulation below. Compute the per cent for the distribution of the budgeted amount for the next year, and the per cent that the expense of China and Glassware is of the income for each department.

Department	*Gross Income*	*China and Glassware Expense*	*Per Cent of Expense*	*Per Cent of Income*
Rooms	\$141,857 50	\$ 269 53	....	....
Restaurant	59,626.90	1,252 16	....	....
Coffee Shop	33,587.45	335 87	....	....
Beverages	9,061.65	423 18	....	....
	\$244,133 50	\$2,280.74	100 00%	....

2. The following budget is that of an estimated operating statement.

		Per Cent of Total Sales
Net sales:		
Class A	\$2,000,000	....
Class B	200,000	....
Class C	250,000	....
Class D	50,000	....
	\$2,500,000	100 00%

		Per Cent of Sales
Production costs:		
Class A	$1,200,000	....
Class B	140,000	....
Class C	162,500	....
Class D	40,000	....
	$1,542,500	....
		Per Cent
Gross margin	$ 957,500	....
Selling:		
Sales administration	$ 50,000	....
General sales department expense	12,500	....
Special promotion, etc	12,500	....
District operating expense	400,000	....
Advertising A	100,000	....
Advertising B	12,500	....
Advertising C	25,000	....
Selling cost	$ 612,500	....
Net margin	$ 345,000	....

Calculate: (*a*) the per cent of net sales in each class, as compared with total net sales; (*b*) the per cent of production cost in each class, based on sales of each class; (*c*) the per cent that selling cost is of total net sales; (*d*) the per cent that the net margin is of total net sales.

3. The following is the budget for the Water Department of a municipality. Find the per cent that each budget expenditure is of the total for the department.

	Amount	*Per Cent*
Pump station and filter plant salaries	$17,300.00	....
Office salaries and expenses	4,600.00	....
Chemicals, filter plant	1,000 00	....
Power—pump station and filter plant	15,000 00	....
Light, heat, and supplies	3,000 00	....
Water service	3,000 00	....
Meters and installation	6,000 00	....
Water main extensions and fire hydrants	3,000 00	....
Motor truck repairs	150 00	....
Interest on outstanding warrants	4,246 00	....
Total	$57,296 00	100 00%

4. Compute the increase or decrease and the per cent of increase or decrease in the following comparative budget.

Public Buildings and Utilities	*This Year*	*Last Year*	*Inc.*	*Dec.*	*% Inc.*	*% Dec.*
City hall engineers and janitors	$ 5,060	$ 4,284	....	....	....	....
City hall fuel and supplies	4,000	2,216	....	....	....	....
City hall maintenance and repairs	1,000	850	....	....	....	....

Public Buildings and Utilities	*This Year*	*Last Year*	*Inc.*	*Dec.*	*% Inc.*	*% Dec.*
Detention hospital repairs...	$ 300	$ 400				
Detention hospital light and fuel....................	900	700				
Park light and fuel..........	1,200	1,050				
Septic tank electric power....	600	600				
Septic tank repairs...	100	100				
Incinerator fuel and light....	1,000	1,000				
Electric lighting—streets, alleys....................	14,500	14,000				
Interest on warrants..... ..	2,284	2,170				
Contingent fund	5,000	4,036				
Detention hospital insurance		433				
Library insurance		281				
Park insurance		104				
	$35,944	$32,224				

Profits based on sales. In the income statement, it is customary to base all percentage calculations on sales. With sales equalling 100%, cost of sales, overhead, and net profit are expressed as per cents of sales. Overhead expenses are those incurred in operating a business—such as salaries and wages, rent, heat, light and power, depreciation, taxes, insurance, advertising, telephone, postage, and so forth. In marking goods bought for resale, these expenses must be taken into consideration. A few items of overhead expense do not fluctuate, but many of them have a fairly constant ratio to gross sales. The merchant determines the ratio of overhead expenses to sales from his own experience and that of others engaged in similar businesses. This per cent of cost of doing business plus the per cent of profit decided upon deducted from 100% determines the per cent which the cost of goods plus freight and drayage bears to the selling price.

<table>
<tr><td colspan="3">Sales = 100%</td></tr>
<tr><td>Invoice Price plus Freight and Cartage
75%</td><td>Overhead
15%</td><td>Profit
10%</td></tr>
<tr><td>Cost of Sales
75%</td><td colspan="2">Gross Profit on Sales
25%</td></tr>
</table>

Example

If overhead charges amount to 15% of sales, and a profit of 10% on sales is desired, what is the selling price of an article with an invoice cost of $21.00 and freight and cartage of $1.50?

Solution

$$15\% + 10\% = 25\%$$
$$100\% - 25\% = 75\%$$
$$\$21.00 + \$1.50 = \$22.50, \text{ the cost.}$$
$$\$22.50 \div 75\% = \$30.00, \text{ the selling price.}$$

Verification

$$25\% \text{ of } \$30.00 = \$7.50, \text{ the overhead and profit.}$$
$$\$30.00 - \$7.50 = \$22.50, \text{ the cost.}$$

Problems

1. An article that cost $15.00 was sold for $20.00. What is the profit per cent on the selling price?

2. With an overhead expense of 20%, what per cent of profit on sales is made by selling for $1.50 articles that have an invoice cost of $1.00?

3. What is the per cent of gross profit on sales in Problem 2?

4. How much must the article in Problem 2 be reduced to sell at cost? What per cent is this of the marked price?

5. A merchant sold an article for $12.00 and made a profit of $12\frac{1}{2}\%$ on the selling price. What was his profit in dollars?

6. Find the per cent of reduction of marked price to produce cost.

	Cost	*Marked Price*	*Per Cent Reduction*
a.	$ 20	$ 25	....
b.	2 50	2 75	....
c.	1 00	1 20	....
d.	03	.05	....
e.	3 00	6 00	....
f.	.40	.50	....
g.	09	.12	....
h.	15.00	25 00	....

7. Find the per cent of profit on the selling price.

	Cost	*Selling Price*	*Per Cent Profit on Selling Price*
a.	$ 1.00	$ 1 20	....
b.	10.00	15.00	....
c.	.60	.75	....
d.	3.50	7.00	....
e.	6.00	8 00	....
f.	150 00	175 00	....
g.	16.00	24 00	....
h.	75 00	125 00	....

8. The factory price of an automobile is $1,300. Freight charges from factory to dealer are $65.00. If the dealer's overhead is 20% and he expects a net profit of 15% on sales, what should be the selling price of the automobile?

9. A furniture dealer bought a shipment of 20 chairs at $30.00 each. He marked them to sell at a profit of 40% on cost. The entire shipment was sold

in the fall clearance sale at 25% reduction from marked price. What was the profit or loss?

10. Complete the following:

	Cost	Selling Price	% on Cost	% on Selling Price
a.	$ 4 00	$ 6 00	....	....
b.	15.00	25 00	....	....
c.	.16	.20	....	....
d.	.04	.08	....	....
e.	.08	.10	....	....
f.	5 00	7 00	....	....
g.	500 00	750 00	....	....
h.	24.00	32 00	....	....

11. The invoice price of an article is $12.00. Freight is 75 cents. It costs 18% to do business and you desire a net profit of 10% on sales. What is the selling price of the article?

12. If the invoice cost is $28.00, freight $2.00, overhead 25%, net profit on sales 15%, what is the selling price?

13. A stock of merchandise valued at $8,750.00 was damaged by fire and water. The loss was estimated to be 25%. Find the value of the damaged merchandise.

14. A merchant's overhead, or cost of doing business, is $22\frac{3}{4}$%. He desires to make a net profit of $7\frac{1}{4}$%. What will be the selling price of an item that cost this merchant $4.90?

15 Merchandise is bought for $3.50 less 25% and sold at $3.50 net. What is the rate per cent of profit?

16. A tea and coffee merchant blends a 40¢ tea with a 70¢ tea in the ratio of 2 to 1. If the blend is sold at 65¢ a pound, what is the rate per cent of profit on cost?

17. A chair manufacturer finds the cost of material in a certain type chair to be $7.50. Manufacturing cost (labor and overhead) is $14.80. Selling and administrative expenses are 20% of sales. What is the manufacturer's price for this chair if he desires to net 10% on the selling price?

18. A clerk was ordered to mark a lot of suits so as to make a profit of 20% after allowing 5% discount for cash. By mistake he marked the suits $24.75 each, which resulted in a loss to the clothier of 8%. At what price should the suits have been marked?

Marking goods. Merchants frequently indicate the cost price and the selling price on each article. The buyer may use the cost price marking for comparison with current quotations; slow sellers may be checked for desirability of reducing the selling price; inventory of stock may be taken at cost; and, under special systems of accounting, a daily record of cost of sales is achieved.

In order to conceal the cost price from the customer, a set of symbols is used, interpretation of which depends upon a knowledge of the key to the letters or characters. Any word or phrase of ten

letters or any ten arbitrary characters may be used as the key. An extra letter or character is used to prevent repetition of a letter, and this extra letter or character is called a *repeater*. The repeater makes it more difficult for a stranger to decipher the marks. The word or phrase used must not contain the same letter twice. Otherwise the same letter will represent two different numbers.

If the cost and the selling price are both written on the same tag, the cost price is usually written below, and the selling price above, a horizontal line. If both cost and selling price are marked, a separate key is used for each.

Example

Use the word "blacksmith" with repeater "w" as the selling key, and the phrase "pay us often" with repeater "x" as the cost key, and mark an article to sell at $6.50 with a cost of $4.25.

Solution

b l a c k s m i t h	Repeater
1 2 3 4 5 6 7 8 9 0	w

p a y u s o f t e n	Repeater
1 2 3 4 5 6 7 8 9 0	x

$$\frac{\text{S.kh}}{\text{U.as}}$$

Example

With the same keys, mark an article to sell at $9.55, with a cost of $7.00.

Solution

$$\frac{\text{T.kw}}{\text{F.nx}}$$

The following are examples of key words and phrases:

Blacksmith	Buy for cash
Charleston	Black horse
Buckingham	Cash profit
Republican	Pay us often
Authorizes	Our last key

Problems

1. Using "Charleston" as the key word and "x" as the repeater, indicate the following costs:

a. $5.56	*d.* $6.62	*g.* $6.20	*j.* $1.44	*m.* $15.00
b. $6.50	*e.* $7.50	*h.* $5.00	*k.* $26.50	*n.* $2.35
c. $2.45	*f.* $12.50	*i.* $.25	*l.* $12.47	*o.* $1.60

2. Use as the cost key "pay us often" and repeater "w," as the selling key "authorizes" and repeater "x," and show markings for the following:

	Cost	Selling Price
a.	$ 2.25	$ 3.50
b.	1.15	1.50
c.	1.25	1 65
d.	23.50	29.50
e.	.65	.90

3. If the selling key is "Bridgepost" with repeater "w," and the cost key is "Cumberland" with repeater "x," write in figures the prices given in the following:

a. $\frac{\text{B.dt}}{\text{C.ud}}$	*e.* $\frac{\text{Bg.ow}}{\text{Cu.dx}}$
b. $\frac{\text{R.pg}}{\text{U.xd}}$	*f.* $\frac{\text{It.ww}}{\text{Ux.dx}}$
c. $\frac{\text{Be.ot}}{\text{Cb.dx}}$	*g.* $\frac{\text{P.et}}{\text{R.bd}}$
d. $\frac{\text{I.tw}}{\text{U.ed}}$	*h.* $\frac{\text{R.dt}}{\text{C.rd}}$

Commissions. The commission business in this country is largely the result of our industrial and commercial development. Economic conditions demand that there shall be agents who shall represent either the buyer or the seller. The compensation paid the agent for his services is called a commission. The principles of percentage apply in commission.

The person who transacts business for another is the *agent,* and the one for whom the business is transacted is the *principal.* The fee, usually a per cent of the dollar volume of the transaction, is the *commission.*

Problems

1. An agent sells oil for $3,475.00 at 3½% commission. What is the amount of the commission?

2. A merchant buys goods through an agent at a cost of $275.00. The agent charges 2½% commission. What is the total cost of the goods to the merchant?

3. An agent sells a consignment of merchandise for $1,824, retaining his commission of 3%. How much does he remit to his principal?

4. If $302.75 was charged for selling $8,650.00 of merchandise, what was the rate of commission?

5. A realtor's fee for selling a house and lot was $150.00. If the rate was 2%, what was the amount received by the principal?

6. An agent's commissions for one week were $216.80. If his sales were $10,840.00, what rate did he charge?

7. The invoice price on a shipment of merchandise was $1,283.38, including agent's commission. If the agent's rate was 3%, what was the commission?

8. The proceeds of a sale received by the principal were $828.78. The commission deducted by the agent was $43.62. What was the rate?

9. Find the net proceeds of the following:

ACCOUNT SALES

Boston, Mass., Oct. 5, 19—

Sold for Account of

Friends Milling Co., Friendsville, Minn.

By Puritan Brokerage Company

19—				
Aug.	4	350 bbls. Flour @ $4.51	....	
	24	175 bbls. Flour @ 4.48	....	
Sept.	6	320 bbls. Flour @ 4.50	....	
	19	60 bbls. Flour @ 4.53	....	
Oct.	1	30 bbls. Flour @ 4.52	....	
		Total Sales		....
		Charges		
Aug.	1	Freight	$420.60	
	4	Cartage	26.50	
Oct.	1	Storage	22.90	
	5	Commission @ 3%	....	
		Total Charges		....
		Net Proceeds		....

10. The manufacturing cost of a certain type machine is $300.00. The manufacturer wishes to catalog this machine at a list price that will net a profit of 25% on sales after allowing a dealer's discount of 25% and agent's commission of $16\frac{2}{3}\%$. Find the catalog list price.

CHAPTER 6

Commercial Discounts

Cash discount. Cash or time discount is a deduction for immediate payment, or for payment within a definite time. The deduction is a certain per cent of the invoice.

The expression "Terms: 2/10, 1/30, n/60" means that 2% of the invoice price may be deducted by the purchaser if payment is made within 10 days of the date of the invoice, that 1% may be deducted if the invoice is paid within 30 days from the date of the invoice, and that the invoice is due in 60 days without discount. In some cases notice is given to the effect that interest at a specified rate will be charged after the due date.

The acceptance of a cash discount is usually of advantage to the purchaser. The following table indicates the annual interest rates to which the usual cash discounts are equivalent:

$\frac{1}{2}$%	10 days, net 30 days	= 9 per cent a year
1%	10 days, net 30 days	= 18 per cent a year
$1\frac{1}{2}$%	10 days, net 30 days	= 27 per cent a year
2%	10 days, net 30 days	= 36 per cent a year
2%	10 days, net 60 days	= 14.4 per cent a year
2%	30 days, net 4 months	= 8 per cent a year

The rate per cent a year is calculated by taking the number of days between the discount date of payment and the end of the credit period, dividing the number of days in a year (360) by this number, and multiplying the quotient by the rate of discount under consideration.

$$\frac{\text{360 Days}}{\text{Number of Days Between Discount Date and End of Credit Period}} \times \text{Rate of Discount} = \text{Equivalent Annual Interest Rate}$$

Problems

1. Find the equivalent annual interest rate for the following terms:

2%	30 days, net 60 days
3%	10 days, net 30 days
3%	30 days, net 60 days
3%	10 days, net 4 months

2. To pay an invoice of $1,500, with terms 2/10, n/30, the purchaser borrowed the money at 6% in order to take advantage of the 2% discount. What benefit did he secure by borrowing the money?

3. A merchant was able to obtain 5% discount on an invoice of $720 by borrowing the money at the bank for 90 days at 6% interest. How much was he able to save?

Trade discount. Mercantile or trade discounts are reductions from list prices, or from the amount of the invoice without regard to time of payment. By offering different rates of trade discounts to wholesalers and retailers, the manufacturer can send the same catalog to both classes of customers. Revised discount sheets are issued as prices fluctuate, but the same catalog may be used a year or more because the list prices are fixed.

Rules of percentage are applied in commercial discounts:

Invoice price	=	base
Per cent of discount	=	rate
Discount	=	percentage

Several discounts are sometimes given. These are known as *chain discounts* or *a series of discounts.*

The order in which the discounts are deducted will not affect the result; thus, a selling price stated as list price "less 10%, 20%, and 5%" is the same as a selling price stated as list price "less 5%, 20%, and 10%." This is shown in the following example, in which $100.00 is used as the base:

Example

$100.00 × .10	$10 00	$100.00 × .05	$ 5 00
$100.00 − $10.00	$90 00	$100.00 − $5.00	$95 00
$ 90.00 × .20	$18 00	$ 95.00 × .20	$19 00
$ 90.00 − $18.00	$72 00	$ 95.00 − $19.00	$76.00
$ 72.00 × .05	$ 3.60	$ 76.00 × .10	$ 7.60
$ 72.00 − $3.60	$68 40	$ 76.00 − $7.60	$68 40
$100.00 − $68.40	$31.60	$100.00 − $68.40	$31.60

The total discount in each case is $31.60.

The dollar amount of discount determined from a series of rates is not the same as the amount of discount determined from a single rate equal to the sum of the series of rates. The sum of the series of rates is 35%; 35% of $100.00 is $35.00, whereas the correct discount is $31.60.

Single discount equivalent to a series.

First method. To find the single discount that is equivalent to a series of discounts, subtract one of the discounts from 100%. Use the remainder as a new base. Multiply it by the second discount, and deduct the product. Use this remainder as a new base.

Compute each discount successively, proceeding as before. The difference between 100% and the last result will be the single discount.

Example

What single discount is equivalent to a series of discounts of 20%, 10%, and $8\frac{1}{3}$%?

Solution

100% − 20%	80%
80% × 10%	8%
80% − 8%	72%
72% × $8\frac{1}{3}$%	6%
72% − 6%	66%
100% − 66%	34%

Explanation. 100% represents the invoice price. 20%, or $\frac{1}{5}$ of 100%, equals 20%, which subtracted from 100% leaves 80%; 10%, or $\frac{1}{10}$ of 80%, equals 8%, which subtracted from 80% leaves 72%; $8\frac{1}{3}$%, or $\frac{1}{12}$ of 72%, equals 6%, which subtracted from 72% leaves 66%. 100%, the invoice price, less 66%, the selling price, leaves 34%, the single discount.

Second method. To find the single discount that is equivalent to a series of discounts, subtract each single discount from 100% and find the product of the remainders. Subtract the final product from 100%, and the remainder is the single discount equivalent to the series of discounts.

Example

What single discount is equivalent to the series 20%, 10%, and 5%?

Solution

100%	100%	100%
20%	10%	5%
80%	90%	95%

$$.80 \times .90 \times .95 = .684, \text{ or } 68.4\%$$
$$100\% - 68.4\% = 31.6\%, \text{ the single discount}$$

A short method. To find the single discount that is equivalent to any two discounts, subtract the product of the discounts from the sum of the discounts.

Example

What single discount is equivalent to discounts of 20% and 20%?

Solution

$$20\% + 20\% = 40\%$$
$$20\% \times 20\% = 4\%$$
$$40\% - 4\% = 36\%$$

To find the net price. To find the net price, the list price and discounts being given: Reduce the discount series to a single dis-

count, multiply the invoice price by this single discount, and deduct the result from the invoice price.

Example

What is the net price of an invoice of $600.00, less 30%, 20%, and 10%?

Solution

100%	100%	100%
30%	20%	10%
70%	80%	90%

.70 × .80 × .90	.504, or 50.4%
100% − 50.4%	49.6%, the rate of discount
$600.00 × 49.6%	$297.60, the discount
$600.00 − $297.60.	$302.40, the net price

If the amount of discount is not desired, the net price may be found as follows:

$600.00 × 50.4% = $302.40

Problems

1. In each of the following, calculate by the short method the single discount that is equivalent to the series:

(*a*) 10% and 5%. (*c*) 40% and 5%. (*e*) 35% and 10%.
(*b*) 20% and 5%. (*d*) 15% and 10%. (*f*) 30% and 20%.

2. In each of the following, find the net price:

(*a*) $350.00, less 10%, 10%, and 5%. (*c*) $480.00, less 20%, 10%, and 5%.
(*b*) $500.00, less $33\frac{1}{3}$%, 5%, and $2\frac{1}{2}$%. (*d*) $1,200.00, less 5%, $2\frac{1}{2}$%, and 1%.
(*e*) $900.00, less 50%, 20%, and 5%.

3. The list price of an invoice is $750.00, with discounts of 10%, 5%, and $2\frac{1}{2}$%. The terms of the invoice are: 2/10, 1/30, and n/60. What amount will be necessary to pay the invoice: (*a*) within the 10-day period; (*b*) within the 30-day period?

4. *B* purchases merchandise listed at $3,500.00, less 20% and 25%. He sells this merchandise at the same list price, less 15%, 10%, and 5%. Does he gain or lose, and what amount?

5. A dealer offers merchandise at a list price of $5,000.00, less discounts of 25%, 10%, and 10%. Another dealer offers the same merchandise at a list price of $4,800.00, less discounts of 20%, 15%, and 5%. Which is the better offer, and by what amount?

6. Which is the better offer, and by what amount, on an invoice of $425.00: (*a*) 30%, 20%, and 10%; or (*b*) a single discount of 50%?

7. The list price of an item is $24.00. If bought at that price less $33\frac{1}{3}$% and 10%, and then sold at the same list price less 20% and 5%, what is the profit?

8. The net cost of an invoice of merchandise was $1,200.00. What was the list price, if the cost was 25% and 20% off list?

9. If the list price is $400.00, and the net price is $380.00, what is the single rate of discount?

10. What single discount is equivalent to 25%, 20%, and 12½%?

Transportation charges on discount invoices. In some cases transportation charges are paid by the seller; in other cases, by the purchaser. If the purchaser is to pay the transportation charge, and as a matter of convenience the seller prepays it, the seller adds the charge to the invoice. The purchaser is not entitled to cash discount on the added charge.

If a shipment is made "freight allowed," the discount should be figured after the deduction for freight; otherwise, it would be equivalent to taking discount on the transportation charge.

Problems

1. An invoice of books amounts to $4.85, and parcel post charges are 79 cents, a total of $5.64. If terms are 2/10, what is the discount if paid within the 10-day period?

2. Complete the following invoice:

6 doz.	Items	@ $6.65	....
24 doz.	Articles	@ .45	....
6¼ doz.	Items	@ .47½	....
3 only	Items	@ 2.34	....
			....
	Less 15%		....
	Less freight allowance 525 lbs. @ .45½ cwt.		....
Net			....

If the terms of the above invoice are 1/10, n/30, what will be the discount if paid within 10 days?

3. An invoice for paper, freight allowed, amounted to $1,754.50. The freight bill paid by the purchaser was $238.54. If 2% discount was allowed for payment within 10 days, what was the amount of the check?

Anticipation. In retail business, invoices for purchases often have *dating terms*. The terms may be 2/10, 90 days extra. If the merchandise is received within 10 days and checked by the receiving department, the purchaser will deduct 2% cash discount, and an anticipation discount on the balance computed at 6% (usually) for 90 days, which is equivalent to an additional discount of 1½%.

Another case is that of spring purchases of fall merchandise invoiced 2/10, November 1 dating. An invoice with these terms may be discounted 2% if paid before November 11, and if paid July 15 would be subject to anticipation discount for 119 days at the customary rate, say, 6%.

Example

An invoice billed April 2, for $3,250.75, terms 2/10, Nov. 1 dating, freight allowed, was paid April 28. Freight paid by the purchaser was $132.48. What was the amount of the check?

Solution

Invoice.	$3,250.75
Less freight.	132 48
	$3,118 27
Less discount, 2%	62 37
	$3,055 90
Less anticipation, 6% for 197 days.	100 34
Amount of check	$2,955 56

Problems

1. An invoice for $21.25 dated Dec. 28, terms 2/10, Feb. 26, was paid Jan. 7. What was the amount of the check?

2. What is the anticipation on an invoice for $475.50, dated June 10, terms 2/10, 90 days extra, if paid June 25?

3. Find the amount earned by paying an invoice for $1,275.25, dated July 12, terms 2/10, Oct. 1 dating, on July 28.

CHAPTER 7

Simple Interest

Definition. Interest, as commonly defined, is a payment for the use of borrowed money or credit. This payment depends upon the rate per cent charged and upon the length of time for which interest is calculated. The sum loaned or the amount of credit used is the principal. The number of hundredths of the principal that is taken is the rate, which is usually expressed as a per cent. The principal, with the interest added, is called the amount.

Short method of calculating. There are a great many methods of computing interest, each of them possessing more or less merit. However, with the accountant the chief consideration is not how many methods there are, but rather how accurately and how quickly he can solve a problem in interest.

The computation of the product of principal, rate, and time is the shortest method when the time is full years or fractional parts of a year, such as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, any number of 10ths, and so forth; otherwise, the operation may be shortened by taking advantage of aliquot parts, multiples and fractions, cancellation, and so forth.

The following principles and methods of computing interest are quick and accurate when the rate is 6%.

Sixty-day method. To find the interest at 6% for:

6 days, point off 3 additional places to the left of the decimal point in the principal.
60 days, point off 2 additional places to the left of the decimal point in the principal.
600 days, point off 1 additional place to the left of the decimal point in the principal.
For 6,000 days, the interest will be the same as the principal.

Example

Find the interest on $256.75 for 6 days at 6%.

Solution

Pointing off 3 places to the left of the decimal point in the principal gives .25675, or 26¢.

Example

Find the interest on $345.65 for 36 days at 6%.

Solution

Point off 3 additional places to the left of the decimal point in the principal, and multiply by 6. The answer is \$2.07.

For rates other than 6%, see adjustments on page 80.

Problems

Find the interest at 6% on:

1. \$180.00 for 60 days.
2. \$150.00 for 54 days.
3. \$262.50 for 24 days.
4. \$32.75 for 36 days.
5. \$65.50 for 12 days.
6. \$26.50 for 18 days.
7. \$752.25 for 6 days.
8. \$15.80 for 54 days.
9. \$75.40 for 30 days.
10. \$12.85 for 24 days.

Method using aliquot parts.

Example

Find the interest on \$275.84 for 124 days at 6%.

Solution

\$2	75 84	=	interest for 60 days
2	75 84	=	interest for 60 days
	18 38	=	interest for 4 days
\$5	70 06	=	interest for 124 days

Explanation. Pointing off 2 decimals, as indicated by the vertical line, gives the interest for 60 days. Double this to find the interest for 120 days. Four days' interest is $\frac{1}{15}$ of 60 days' interest. The sum, \$5.70, is the interest for 124 days.

Example

Find the interest on \$754.90 for 137 days at 6%.

Solution

\$ 7	54 90	=	interest for 60 days
7	54 90	=	interest for 60 days
1	50 98	=	interest for 12 days
	62 90	=	interest for 5 days
\$17	23.68	=	interest for 137 days

Explanation. Pointing off 2 decimal places gives the interest for 60 days. Double this to find the interest for 120 days. Twelve days is $\frac{1}{5}$ of 60 days; therefore, the interest for 12 days is $\frac{1}{5}$ of 60 days' interest, or \$1.5098. Five days is $\frac{1}{12}$ of 60 days, and the interest is $\frac{1}{12}$ of \$7.5490, or \$0.629. The sum, \$17.24, is the interest for 137 days.

After a little practice, any number of days can be resolved into 6- or 60-day periods and easy fractions thereof.

Example

Find the interest on \$247.64 for 8 days at 6%.

Solution

$$\begin{array}{rl} \$|247.64 & = \text{interest for 6 days} \\ |082.54 & = \text{interest for 2 days} \\ \hline \$|330.18 & = \text{interest for 8 days} \end{array}$$

Explanation. To find the interest for 6 days, point off 3 decimals, as indicated by the vertical line. Two days' interest is $\frac{1}{3}$ of 6 days' interest. The answer is, therefore, 33¢.

For rates other than 6%, see adjustments on page 80.

Problems

Find the interest at 6% on:

1. $286.75 for 9 days.
2. $189.22 for 8 days.
3. $256.35 for 27 days.
4. $178.56 for 39 days.
5. $38.29 for 40 days.
6. $175.82 for 34 days.
7. $38.95 for 19 days.
8. $47.56 for 17 days.
9. $29.10 for 2 days.
10. $1,286.75 for 21 days.

The cancellation method. The cancellation method may be used to advantage in many interest calculations, especially in those having fractional rates and rates other than 6%.

Example

Find the interest on $750.00 for 45 days at 5%.

Solution

$$\frac{\overset{125}{\cancel{750}} \times \overset{15}{\cancel{45}} \times \overset{.01}{\cancel{.05}}}{\underset{4}{\cancel{12}} \times \underset{6}{\cancel{30}}} = \frac{18.75}{4} = 4.687, \text{ or } \$4.69$$

Explanation. Writing below the line 12 times 30, instead of 360 days, facilitates cancellation.

Problems

Find the interest, by the cancellation method, on:

1. $840.00 for 12 days at 2%.
2. $320.00 for 15 days at 4%.
3. $160.80 for 16 days at 5%.
4. $275.75 for 74 days at 6%.
5. $112.50 for 85 days at 4%.
6. $284.00 for 34 days at 6%.
7. $368.00 for 56 days at 5%.
8. $775.14 for 79 days at 5%.
9. $250.00 for 91 days at 6%.
10. $500.00 for 102 days at 4%.

Example

Find the interest on $345.75 for 90 days at $4\frac{1}{2}$%.

Solution

$$\frac{345.75 \times \overset{3}{\cancel{90}} \times .09}{\underset{4}{\cancel{12}} \times \cancel{30} \times 2} = \frac{31.1175}{8} = \$3.889, \text{ or } \$3.89$$

Problems

Find the interest, by the cancellation method, on:

1. \$360.80 for 38 days at $4\frac{1}{2}\%$.
2. \$312.32 for 45 days at $4\frac{3}{4}\%$.
3. \$1,000.00 for 40 days at $5\frac{1}{2}\%$.
4. \$1,600.00 for 75 days at $4\frac{1}{4}\%$.

Dollars-times-days method, 6%. This method is rapid, and is particularly valuable when a calculating machine is used. It is a modification of the cancellation method, where 6% and 360 days are two of the factors. Thus:

$$\frac{\$ \times \text{Days} \times .0\cancel{6}}{\cancel{360}\atop 6{,}000}$$

Assume that there are no other items that can be cancelled. The number of dollars is multiplied by the number of days, and the product divided by 6,000. Any number may be divided by 6,000 by pointing off 3 decimals, and dividing the resultant number by 6.

Example

Find the interest on \$256.50 for 28 days at 6%.

Solution

Multiply the number of dollars by the number of days, *point off 3 decimal places in addition to the number of decimal places in the principal, then divide by 6.*

```
     $256 50
          28
   6)7 182 00
     1.197      or $1.20, the interest
```

This method may be used for any rate by adding to or subtracting from the interest computed at 6%, the fractional part thereof that the specified rate is greater or less than the 6% rate.

For 8%, increase the interest by $\frac{1}{3}$ of the amount computed at 6%.
For 7%, increase the interest by $\frac{1}{6}$ of the amount computed at 6%.
For 5%, decrease the interest by $\frac{1}{6}$ of the amount computed at 6%.
For 4%, decrease the interest by $\frac{1}{3}$ of the amount computed at 6%.
For $4\frac{1}{2}\%$, decrease the interest by $\frac{1}{4}$ of the amount computed at 6%.

The above adjustments may be used with any of the 6% methods in solutions in which the rate is more or less than 6%.

Problems

Find the interest on the following:

1. \$275.12 for 73 days at 5%.
2. \$132.86 for 28 days at 8%.
3. \$280.60 for 70 days at 4%.
4. \$138.42 for 28 days at $4\frac{1}{2}\%$.
5. \$276.95 for 17 days at 8%.
6. \$640.64 for 56 days at 7%.

Interchanging principal and time. Under the 60-day method, the computations may often be shortened by interchanging the principal and the time.

Example

Find the interest on $6,000.00 for 31 days at 6%.

Solution

Interchanging the principal and the time, the problem becomes that of finding the interest on $31.00 for 6,000 days. Apply the 6%, 60-day method, and the interest is found to be $31.00, since the interest is equal to the principal when the rate is 6% and the time is 6,000 days.

Problems

Find the interest on the following:

1. $2,400.00 for 23 days at 6%.
2. $3,600.00 for 7 days at 6%.
3. $6,000.00 for 156 days at 6%.
4. $3,000.00 for 193 days at 6%.
5. $4,500.00 for 38 days at 6%.
6. $4,200.00 for 41 days at 6%.

Exact or accurate interest. Exact or accurate interest is that which is obtained when a year is taken as 365 days. For full years, all methods of computing interest give the same result— a certain per cent of the principal; hence the results differ only when fractional parts of a year are used.

Example

Find the exact interest on $1,200.00 for 93 days at 6%.

Solution

The cancellation method previously explained is the method used, as it is probably the most practical.

$$\frac{\overset{240}{\cancel{1200}} \times 93 \times .06}{\underset{73}{\cancel{365}}} = \frac{1,339.20}{73} = \$18.35$$

Problems

Find the exact interest on:

1. $750.00 for 45 days at 6%.
2. $1,200.00 for 68 days at 7%.
3. $1,600.00 for 73 days at 6½%.

Accumulation of simple interest. Simple interest accumulates in like amount each period, if the principal and rate are unchanged.

Symbols. Let i equal the rate of interest, n the number of periods, and P the principal. Then accumulation of simple interest on any sum of money, for any number of periods, may be found as follows:

Example

Find the simple interest on $100.00 for 5 years at 6%.

Algebraic Formula	*Arithmetical Substitution*
$P(1 \times in) = \text{Interest.}$	$100(1 \times .06 \times 5) = 30.$

Solution

$1 \times .06 = .06$, interest on 1 for 1 year at 6%
$.06 \times 5 = .30$, interest on 1 for 5 years at 6%
$.30 \times 100 = \$30.00$, interest on $100.00 for 5 years at 6%

TABLE OF SIMPLE INTEREST

(1) End of Year	(2) Principal	(3) Interest Due Each Year	(4) Total Int. at End of Each Year	(5) Sum Due at End of Year
1	$100 00	$6.00	$ 6.00	$106 00
2	100 00	6.00	12 00	112.00
3	100.00	6.00	18 00	118 00
4	100.00	6 00	24 00	124 00
5	100.00	6 00	30.00	130 00

Problems

1. A man borrows $500.00 for 9 years at 4%. What amount of interest will he pay during this period? Write the formula and solution, as shown in the example above.

2. What is the amount of interest due on $300.00 at the end of 10 years if the rate is 7%? Write the formula and solution, as shown in the example above.

3. Construct a table in columnar form, similar to the table above (omitting column 5), for $400.00 invested for 5 years at 6%.

4. What is the interest accumulation on a debt of $4,270.00 for 8 years at 5% simple interest?

Simple amount. The simple amount is found by adding to the principal the total simple interest. It is the amount due at the end of the stated period.

Example

What is the amount of $100.00 for 5 years at 6%?

Algebraic Formula	*Arithmetical Substitution*
$P + [P(1 \times in)] = \text{Amount.}$	$100 + [100(1 \times .06 \times 5)] = 130.$

Solution

$1 \times .06 = .06$, interest on 1 for 1 year at 6%
$.06 \times 5 = .30$, interest on 1 for 5 years at 6%
$100 \times .30 = 30.00$, interest on 100 for 5 years at 6%
$100 + 30.00 = \$130.00$, amount of $100.00 for 5 years at 6%

The simple amount is shown in column 5 of the table above; hence the construction of a table is omitted here.

Problems

Write formulas and solutions for the following:

1. The amount of a $200.00 note due in 6 years; interest, 5%.

2. The amount due in 6 years on $530.00 at 6%.

3. The amount due on a note for $750.00 with 4% interest. No interest has been paid during a period of 4 years.

Rate. To find the rate, when the principal, interest, and time are given, divide the given interest by the interest on the principal at 1% for the given time.

Example

At what rate will $100.00 produce $24.00 in 4 years?

Algebraic Formula

$$\frac{\text{Interest}}{P(1 \times in)} = \text{Rate of interest.}$$

Arithmetical Substitution

$$\frac{24}{100(1 \times .01 \times 4)} = 6.$$

Solution

$1 \times .01 = .01$, interest on 1 at 1% for 1 year
$.01 \times 4 = .04$, interest on 1 at 1% for 4 years
$100 \times .04 = 4.00$, interest on 100 at 1% for 4 years
$24 \div 4 = 6$, or 6%, the rate of interest

Problems

Write formulas and solutions for each of the following:

	Principal	*Interest*	*Time*	*Rate*
1.	$ 400 00	$ 48.00	3 years	....
2.	2,000.00	500.00	5 years	....
3.	800.00	336 00	6 years	....
4.	300 00	126.00	7 years	....
5.	150.00	40.50	6 years	....

Time. To find the time, when the principal, interest, and rate of interest are given, divide the interest by the product of the principal and the given rate for one year.

Example

In what time will $100.00 invested at 6% produce $24.00 interest?

Algebraic Formula

$$\frac{\text{Interest}}{P(1 \times in)} = \text{Time.}$$

Arithmetical Substitution

$$\frac{24}{100(1 \times .06 \times 1)} = 4.$$

Solution

$1 \times .06 = .06$, interest on 1 at 6% for 1 year
$100 \times .06 = 6$, interest on 100 at 6% for 1 year
$24 \div 6 = 4$, the number of years

Problems

	Principal	*Interest*	*Rate*	*Time*
1.	\$1,000 00	\$240 00	6%	....
2.	750 00	90 00	4%	....
3.	360 00	81 00	$4\frac{1}{2}$%	....

Present worth. The present worth of a debt, due at some future time, without interest, is the sum which must be invested now in order to produce the specified amount at the end of the period. Thus, since \$1 invested for 5 years at 6% will amount to \$1.30, the amount that must be invested now at 6% to produce \$1 at the end of 5 years is 1.00/1.30, or \$.7692.

To find the present worth of a sum, multiply the sum by the present worth of \$1.00 for the given time.

Example

What is the present worth of a note for \$100.00, due in 5 years, without interest, money being worth 6%?

Algebraic Formula | *Arithmetical Substitution*

$$P\left(\frac{1}{1+(1\times in)}\right) = \text{Present worth.} \qquad 100\left(\frac{1}{1+(1\times .06\times 5)}\right) = 76.92.$$

Solution

$1 \times .06 =$ .06, interest for 1 year on 1
$.06 \times 5 =$ .30, interest for 5 years on 1
$1 + .30 =$ 1.30, amount of 1 for 5 years
$1 \div 1.30 =$ .7692, present worth of 1 for 5 years
$100 \times .7692 =$ \$76.92, present worth of \$100.00 for 5 years

Verification

\$76.92 × .06 = \$4.6154, interest for 1 year on present worth
\$4.6154 × 5 = \$23.08, interest for 5 years on present worth
\$76.92 + \$23.08 = \$100.00, amount due in 5 years

TABLE OF PRESENT WORTH

(1) *Years*	(2) *Principal*	(3) *Divided by Amount of \$1*	(4) *Equals Present Worth*	(5) *Principal Minus Present Worth Equals Discount*
1	\$100 00	\$1.06	\$94 34	\$ 5 66
2	100 00	1.12	89 29	10.71
3	100.00	1.18	84.74	15 26
4	100.00	1.24	80 65	19 35
5	100.00	1 30	76 92	23 08

Comparison of simple amount and simple present worth. The following comparative chart is presented to illustrate the accumu-

lation of simple interest on a sum and on the present worth of the same sum:

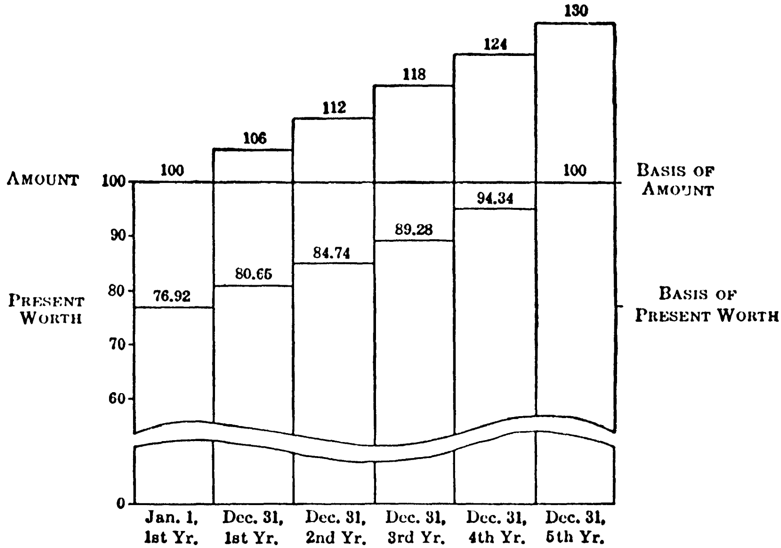


Figure 1.

The amount starts at \$100.00, and accumulates to \$130.00 in 5 years. The present worth starts at \$76.92, and accumulates to \$100.00. The rate of interest is the same in each case, 6%.

Problems

1. What is the present value of a 6-year note for \$650.00, without interest, if money is worth 5%? Write the formula and solution, as shown in the example on page 84.

2. A note for \$3,500.00, without interest, is due in 5 years. What is its present value, money being worth 6%? Write the formula and solution, as shown in the example on page 84.

3. Construct a table in columnar form, similar to the table on page 84, (omitting column 5). Use \$1.00 as the principal, 4 years as the time, and 5% as the interest rate.

4. Construct a comparative chart showing the difference in value of the amount and the present worth of \$400.00 due in 8 years, interest at 5%.

True discount. True discount is the difference between the sum due and its present worth computed on a simple interest basis. (See page 84.)

Example

Find the true discount on a debt of \$100.00 due in 5 years, without interest, money being worth 6%.

Solution

The present worth of the debt is the sum shown in the solution on page 84, or $76.92.

$100.00 − $76.92 = $23.08, the true discount

Refer to column 5 of the Table of Present Worth, page 84, for the method of showing the true discount in columnar form.

Problems

1. What is the true discount on a debt of $750.00 due in 3 years, money being worth 4%? Write the formula and solution.

2. Find the difference between the present value and the face value of a non-interest-bearing note for $500.00, due in 4 years, money being worth 6%. Write the formula and solution.

3. Construct a table in columnar form, similar to the table on page 84, for $1.00 at 4% for 6 years.

4. Find the difference between the true discount and the simple interest on $650.00 for 8 years at 4%.

CHAPTER 8

Bank Discount

Definition. Bank discount is a deduction made from the amount due at maturity on a note or draft, in consideration of its being converted into cash before maturity. If the note does not bear interest, its face value is the amount due at maturity. If the note does bear interest, the amount due at maturity is the face value plus interest on the face value for the period and at the rate specified in the note.

In bank discount, the time is the period from the date of discount to the date of maturity of the note. The date of maturity of a note is the day on which it is due. Notes due a given number of days after date mature after the exact number of days have elapsed. Notes due a given number of months after date mature on the same date so many months hence, except notes made on the 31st and falling due in a 30-day month, which mature on the 30th, and notes made on the 29th, 30th, or 31st of some month and falling due in February, which mature on the last day of February.

Example

A note due 30 days after January 31, will mature on March 2; but if the note is due in one month, it will mature on the last day of the succeeding month, or February 28. If the year should be a leap year, the maturity dates would be March 1 and February 29.

Counting time. In counting time, the usual method is to count the first succeeding day as one day. To illustrate, if a note is given on January 15 for 10 days, the 16th is counted as the first, the 17th as the second, the 18th as the third, and January 25 as the tenth day.

Finding the difference between dates by use of a table. By numbering the days of the year, a calendar may be made for determining the number of days between any two dates. A portion of such a table, and the use made of it, are illustrated on page 88.

May 1	121	Nov. 1	305
2	122	2	306
3	123	3	307
4	124	4	308
5	125	5	309
6	126	6	310
7	127	7	311
8	128	8	312
9	129	9	313
10	130	10	314

The number of days between May 4 and November 9 is found as follows:

The table shows that November 9 is the	313th	day of the year
The table shows that May 4 is the	124th	day of the year
Therefore, the difference is	189	days, the time required

Another form of table is one that shows the number of days from any day of any month to the corresponding day of any other month not more than one year later.

	Jan.	*Feb.*	*Mar.*	*Apr.*	*May*	*June*	*July*	*Aug.*	*Sept.*	*Oct.*	*Nov.*	*Dec.*
January	365	31	59	90	120	151	181	212	243	273	304	334
February	334	365	28	59	89	120	150	181	212	242	273	303
March	306	337	365	31	61	92	122	153	184	214	245	275
April	275	306	334	365	30	61	91	122	153	183	214	244
May	245	276	304	335	365	31	61	92	123	153	184	214
June	214	245	273	304	334	365	30	61	92	122	153	183
July	184	215	243	274	304	335	365	31	62	92	123	153
August	153	184	212	243	273	304	334	365	31	61	92	122
September	122	153	181	212	242	273	303	334	365	30	61	91
October	92	123	151	182	212	243	273	304	335	365	31	61
November	61	92	120	151	181	212	242	273	304	334	365	30
December	31	62	90	121	151	182	212	243	274	304	335	365

Example

A note due August 17 was discounted June 10. What was the term of discount?

Solution

From the table, June 10 to August 10	61 days
August 10 to August 17	7 days
Total	68 days

Banks do not compute time uniformly, but the methods given here are in common use.

NOTE: Tables similar to those above may also be used to good advantage in computing the unexpired time of insurance policies.

Proceeds. The proceeds of a note is the difference between the amount due at maturity and the bank discount.

To find bank discount and proceeds.

Compute the bank discount as simple interest on the amount due at maturity for the unexpired time (term of discount). Deduct the bank discount from the value of the note at maturity to obtain the proceeds.

Example 1

Find the bank discount and the proceeds if a non-interest-bearing note for $420.00 due in 90 days is discounted at 6%.

Solution

Face of note	$420.00
Bank discount, 90 days	6 30
Proceeds	$413.70

Example 2

A note for $780.00 dated May 5, payable in 6 months with interest at 6%, is discounted at 6% on August 3. Find the bank discount and the proceeds.

Solution

Face of note	$780 00
Interest for 6 months at 6%	23 40
Value of note at maturity	$803 40
Bank discount on $803.40 for 94 days at 6%	12 59
Proceeds	$790 81

To find the face of a note when the proceeds, time, and rate of discount are given. Divide the proceeds of the note by the proceeds of $1.00 for the given rate and time.

Example

For what sum must a non-interest-bearing note be drawn, due in 90 days, so that when it is discounted at a bank at 6% per annum, the proceeds will be $537.40?

Solution

$0.015 = bank discount on $1.00 for 90 days
$1.00 − $0.015 = $0.985, the proceeds of $1.00
$537.40 ÷ $0.985 = 545.58 times, or $545.58

Problems

Find the bank discount and the proceeds on:

1. A note for $750.00, due May 30 without interest, and discounted April 16 at 6%.

2. A note for $1,200.00, due December 4 without interest, and discounted Oct. 29 at 6%.

3. A note for $1,500.00, dated October 8 and due in 4 months with interest at 6%, discounted December 1 at 6%.

4. A note for $800.00, dated September 9 and due in 6 months with interest at 7%, discounted November 11 at 6%.

5. A note for $250.00, dated July 11 and due in 90 days with interest at 5½%, discounted September 1 at 6%.

6. *$443.03 was received as the proceeds of a 90-day note discounted at 6%. What was the face of the note?*

7. *For what sum must a 60-day note be drawn in order that the proceeds will be $600.00 when the note is discounted at 6%?*

8. Find the date of maturity, the term of discount, the bank discount, and the proceeds of a 60-day note for $750.00, dated July 8 and discounted July 17 at 5%.

9. Find the date of maturity and the term of discount of a 90-day sight draft, dated May 14, accepted May 17, and discounted June 10.

10. Find the date of maturity, the term of discount, the bank discount, and the proceeds of a note for $650.00, dated Nov. 30, due in 3 months, and discounted Jan. 5 at 6%.

CHAPTER 9

Partial Payments

Part payments on debts. A debtor may by agreement make equal or unequal payments on the principal at regular or irregular intervals. Any partial payment of a note or draft should be recorded on the back of the note or draft.

Methods. There are two methods of applying payments of principal and interest to the reduction of an interest-bearing debt. The method adopted by the Supreme Court of the United States is termed the "United States Rule"; the other method, which has been widely adopted by businessmen, is termed the "Merchants' Rule."

United States Rule. The United States Rule is now a law in many of the states, having been made so either by statute or by court decision.

The court holds that when a part payment is made on an interest-bearing debt, the payment must first be used to discharge the accumulated interest, and what remains of the payment is then applied in cancellation of the principal. If the payment is smaller than the accumulated interest, no cancellation takes place, and the previous principal continues to draw interest until the accumulated payments exceed the accumulated interest.

Example

An interest-bearing note for $1,800.00 dated March 1, 1944, had the following indorsements:

September 27, 1944	$500 00
March 15, 1945	25 00
June 1, 1945	700 00

How much was due September 1, 1945?

Solution

	Yr. Mo. Da.	*Yrs.*	*Mos.*	*Days*
Date of note	1944—3—1			
First payment, $500.00	1944—9—27		6	26
Second payment, $25.00	1945—3—15		5	18
Third payment, $700.00	1945—6—1		2	16
Settlement	1945—9—1		3	0
		1	6	0

Explanation. The time is found by successive subtractions of the first date from the second, the second from the third, and so on. The sum of the different times is equal to the time between the date of the note and the date of settlement.

Face of note, March 1, 1944		$1,800 00
Interest on $1,800.00 at 6% from March 1 to Sept. 27, 6 months and 26 days		61 80
Amount due Sept. 27, 1944		$1,861 80
Deduct payment		500 00
Balance due Sept. 27, 1944		$1,361 80
Interest on $1,361.80 at 6% from Sept. 27 to March 15, 5 months and 18 days, $38.13. As this interest is larger in amount than the payment made at March 15, the interest is not added and the payment is not deducted.		
Interest on $1,361.80 at 6% from Sept. 27 to June 1, 1945, 8 months and 4 days		55 38
Amount due June 1, 1945		$1,417.18
Deduct sum of payments: March 15	$ 25 00	
June 1	700 00	725 00
Balance due June 1, 1945		$ 692 18
Interest on $692.18 at 6% from June 1 to Sept. 1, 1945, 3 months		10 38
Balance due September 1, 1945		$ 702 56

Problems

1. A note for $1,650.00 was dated May 20, 1944. The interest was 6% from date, and the following payments were indorsed:

Sept. 8, 1944	$ 45 00
Dec. 14, 1944	20 00
Feb. 26, 1945	50 00
July 5, 1945	90 00
Nov. 14, 1945	250 00

What amount was due December 17, 1945?

2. A note for $1,200.00 was dated June 20, 1941. The interest was 6% from date, and the following payments were indorsed:

Oct. 2, 1941	$120 40
February 8, 1942	29 50
May 23, 1942	56 40
December 11, 1942	388 75

What amount was due January 23, 1943?

3. A note for $1,000.00 was dated April 10, 1938. The interest was 7% from date, and the following payments were made:

November 10, 1939	$ 80 50
July 5, 1940	100 00
January 10, 1941	450 80
October 1, 1943	500 00

What amount was due January 1, 1944?

Merchants' Rule. Find the amount of the debt (principal and interest) to the date of final settlement, or if the debt runs for more than one year, find the amount to the end of the first year. Deduct from this the sum of all the payments and interest on same to the date of settlement, or to the end of the year. The remainder will be the amount due at the date of settlement, or at the beginning of the next year.

Example

For purposes of comparison, the same problem will be used here as was used to illustrate the United States Rule.

Solution

Face of note, March 1, 1942		$1,800 00
Interest, 1 year at 6% to March 1, 1943		108 00
		$1,908 00
Deduct:		
First payment, September 27, 1942	$500 00	
Interest at 6% to March 1, 1943, 5 months and 4 days	12 83	512 83
Balance due at beginning of second year		$1,395 17
Interest on $1,395.17 at 6%, March 1 to Sept. 1, 1943, 6 months		41 86
		$1,437 03
Deduct:		
Second payment, March 15, 1943	$ 25 00	
Interest at 6% from March 15 to Sept. 1, 1943, 5 months and 16 days...	69	
Third payment, June 1, 1943	700.00	
Interest at 6% from June 1 to Sept. 1, 1943, 3 months	10 50	736 19
Balance due		$ 700 84

The difference of $1.72 between the balance as computed by the Merchants' Rule and the balance as computed by the United States Rule is small, but a much greater difference will occur when the time is long and the amount large.

It is usual to compute the balance due on notes of one year or less by the Merchants' Rule.

Problems

1. A note for $950.00 with interest at 6% was dated Feb. 3, 1943, and had the following indorsements:

March 1, 1943........	$150.00	July 8, 1943	$300.00
June 3, 1943..........	96.00	December 20, 1943....	250.00

What amount was due January 17, 1944?

2. A note for $791.84 with interest at 6% was dated December 14, 1942, and bore the following indorsements:

January 3, 1943	$100 00	July 29, 1943	$324 00
March 16, 1943	240.00	August 3, 1943	20.00

What amount was due November 14, 1943?

3. A note for $1,200.00 dated April 1, 1942, bore interest at 7% and had the following indorsements:

April 12, 1942	$161 08	July 28, 1942	$ 17 90
July 19, 1942	224 14	January 29, 1943	100 25

What amount was due April 1, 1943?

CHAPTER 10

Business Insurance

Kinds of insurance. There are at least 21 kinds of insurance applicable to the ordinary business being done in big cities and as many as 150 kinds of insurance covering all branches of human endeavor.

Policy. An insurance policy is a written contract. The consideration given for the protection promised consists of a premium to be paid in money and the fulfillment by the insured of acts of commission and omission according to the terms and conditions set forth in the policy.

Fire insurance. Fire insurance is guaranty of indemnity for loss or damage to property by fire. Insurance companies are liable for loss or damage resulting from the use of water or chemicals used in extinguishing the fire and from smoke. A fire loss is predicated on the sound value at the time the loss is sustained and not at the time the insurance is written.

Form of policy. With but a few exceptions, fire insurance companies use a State standard-form policy made mandatory by the State in which they operate. The New York State standard form of policy is the one that is generally used, as it embraces nearly all that is contained in other forms. The form attached to the policy is known as a *rider*. The rider form directly applies the insurance to fit the facts and conditions of the particular risk. It also amends the standard form, which is not a contract until completed by descriptions and amendments.

Rates. Probably no phase of insurance interests the businessman more than his insurance rate. Independent rating bureaus operate in different parts of the country. Their business is to inspect and to measure the hazards in terms of rates. Rate schedules are compiled for this purpose. The charges are in the nature of penalties for hazards.

Example

A particular building has been inspected and surveyed by the rater. The degree of municipal and local protection has been measured. This establishes the basic rate. Assume the basic rate to be .40, which is a charge commensurate with the degree of protection and covers all general hazards that cannot be

segregated and measured. The better the city protection, the lower will be the basic rate.

Basic rate		.40
Area: 15,800 sq. ft		.04
(The standard unit area is 1,000 sq. ft., and an additional charge is made for larger areas.)		
Parapet wall deficiency		04
Skylights not standard construction		02
Metal stacks through roof		06
Outside wood cornices, loading docks, and wooden conveyor		.06
Gallery decks used for storage		.03
Occupancy hazard (woodworking mill)		.92
Shavings allowed to accumulate		.05
No cans for collecting waste		.05
No drip pans under machines		05
Floor oil-soaked		05
Total		1 77
Credit for open finish (inside walls)		08
Building rate unexposed		1.69
Exposure:		
From buildings No. 2 and No. 5 at 18 ft	.34	
From building No. 6 at 15 ft	.02	
From office at 23 ft	.05	
From buildings No. 9 and No. 10	07	
Exposure charge		.48
Total building rate		2 17

If this assured would have the parapet wall brought up to the standard requirements, his rate would be reduced .04. By having the shavings removed daily, and by installing waste cans and drip pans under the machines, the rate could be reduced another .15. As a matter of fact, the owner makes his own rate—the rater simply measures the hazards in terms of rates.

To find the premium. Insurance companies charge a certain number of cents or dollars for insuring each $100.00 worth of property. Thus, the insurance rate in the foregoing example is $2.17 for each hundred dollars of insurance carried. If the building is valued at $65,000.00 and is insured for full value, the amount of the premium would be computed as follows:

$2 17	the rate per $100.00 of insurance.
× 6 50	the number of hundred dollars of insurance purchased.
$1,410.50	the premium, or cost of the insurance for one year.

Agent's commission. Local agents of the fire insurance companies are located in almost every city and town. They act as the representatives of the companies, soliciting the business and collecting the premiums. For this service they receive a certain per cent of the premiums.

Example

A store building valued at $10,000.00 was insured for 80% of its value, the rate being $1.25 a hundred. What was the agent's commission if he received 15% of the premium?

Solution

80% of $10,000.00 = $8,000.00, the insured value.
80 × $1.25 = $100.00, the premium.
15% of $100.00 = $15.00, the agent's commission.

Cancellation of policies. Both the insurance company and the insured have the right to cancel an insurance policy at any time. When the policy is canceled by the insurance company, the portion of the premium to be repaid to the insured is determined pro rata.

Example

On April 10, the owner of a building insured his property for one year. The premium was $36.00. On October 20, the policy was canceled by the insurance company. What rebate did the insured receive for the unexpired term?

Solution

The time from April 10 to October 20 is 193 days, expired term of the policy. (See page 88 for table of number of days between dates.)

$\frac{193}{365}$ of $36.00 is $19.04, amount of premium earned.
$36.00 − $19.04 = $16.96, amount of premium returned.

When the policy is canceled by the insured, the amount of premium returned is determined by the "short rate." The short rate is an arbitrary per cent fixed by the insurance companies, and is shown by a table like the one at the top of page 98.

Example

On May 2, a one-year policy was written on a shop. The premium was $38.75. On September 26, the policy was canceled at the request of the insured. What rebate did the insured receive?

Solution

From May 2 to September 26 is 147 days. The table shows that 60% of the premium is to be retained when the policy has been in force 150 days, which is the number of days next higher than 147. Then 40% of the premium will be returned

$38.75 × .40 = $15.50, the return premium.

Coinsurance. This is a form of insurance in which the person who insures his property agrees to carry insurance equal to a certain percentage of the valuation of the property. If he fails to carry that percentage with an insurance company, he (the

SHORT RATE CANCELLATION TABLE

Period exceeding 20 days and not exceeding 25 days, to be the rate of 25 days, and so on up to one year.

Policy in Force	*Per Cent of Annual Prem.*	*Policy in Force*	*Per Cent of Annual Prem.*
1 day	2%	55 days	29%
2 days	4%	60 days or 2 months	30%
3 days	5%	65 days	33%
4 days	6%	70 days	36%
5 days	7%	75 days	37%
6 days	8%	80 days	38%
7 days	9%	85 days	39%
8 days	9%	90 days or 3 months	40%
9 days	10%	105 days	46%
10 days	10%	120 days or 4 months	50%
11 days	11%	135 days	56%
12 days	11%	150 days or 5 months	60%
13 days	12%	165 days	66%
14 days	13%	180 days or 6 months	70%
15 days	13%	195 days	73%
16 days	14%	210 days or 7 months	75%
17 days	15%	225 days	78%
18 days	16%	240 days or 8 months	80%
19 days	16%	255 days	83%
20 days	17%	270 days or 9 months	85%
25 days	19%	285 days	88%
30 days or 1 month	20%	300 days or 10 months	90%
35 days	23%	315 days	93%
40 days	25%	330 days or 11 months	95%
45 days	27%	345 days	98%
50 days	28%	360 days or 12 months	100%

insured) becomes a coinsurer on the loss, in the ratio which his lack of insurance bears to the amount he should have carried.

Illustration of 80% coinsurance clause:

Case 1.	Value of building and contents	$75,000
	Assured should carry 80% of value or	60,000
	Insurance actually carried	45,000
	Loss by fire	10,000
	Paid by insurance company, 75% of loss, or	7,500
	Assured must bear 25% of loss, or	2,500

Insurance carried was only 75% of what assured should have carried to comply with the 80% clause.

Case 2.	Value of property	$10,000
	Insurance required	8,000
	Insurance carried	9,000
	Losses up to $9,000	Paid in full

Case 3.	Value of property	$10,000
	Insurance required	8,000
	Insurance carried	8,000
	Losses exceeding $8,000	
	Face of policy, $8,000, is paid.	
Case 4.	Value of property	$10,000
	Insurance required	8,000
	Insurance carried	5,000
	Losses exceeding $8,000	
	Face of policy, $5,000, is paid.	
	Losses under $8,000	
	Paid in the proportion that $5,000 bears to $8,000, or $\frac{5}{8}$ of the loss.	

Problems

1. What premium must be paid on a policy for $3,760 at $1.50 a hundred?

2. A house worth $12,000 is insured for $\frac{3}{4}$ of its value for three years at $2.35 a hundred. How much is the premium?

3. An agent wrote a policy of $4,500 on a store building at a rate of 85 cents. If the agent's commission was 15%, what was the amount of his commission?

4. Find the amount paid by the insurance company under the 80% coinsurance clause in the following:

	(*a*)	(*b*)	(*c*)	(*d*)
Value of property	$50,000	$75,000	$100,000	$200,000
Insurance carried	40,000	60,000	80,000	80,000
Loss by fire	10,000	45,000	40,000	40,000
Paid by insurance company	....	....	....	....

5. You are presented the following tornado insurance plan and are asked to select one of the four policies and to decide upon whether to insure for one or three years. In your opinion, what policy should be taken and for how long a term? The sound value of the property to be insured is $1,242,000.

Coinsurance	*Amount of Insurance*	*One-Year Rate*	*Premium*	*Three-Year Rate*	*Premium*
(1) None	$ 200,000	.20	....	.50	....
(2) 50%	621,000	.102	....	.255	....
(3) 80%	993,600	.0749	....	.1872	....
(4) 90%	1,117,800	.0678	....	.1695	....

Compute the premiums at the one-year rate and at the three-year rate. Find the average yearly premium on each policy at the three-year rate, and make comparisons in order to determine which policy to accept.

6. A one-year policy on a dwelling was dated June 5. The premium was $42.50. On October 1, the policy was canceled at the request of the insured. Find the amount of return premium.

Use and occupancy insurance. This kind of insurance is protection against loss due to interruption of business by fire or tornado. It is insurance against a loss that is suffered on account of destruction of the property.

The insurance recovery or indemnity is the profit that would have been made if business had not been interrupted and, in addition, the total of expenses that must continue during suspension of business. A business that is not profitable may be so insured in order to recover the continuing expenses.

Generally speaking, use and occupancy insurance insures gross profits plus the salaries of key employees kept on the payroll account. The policy excepts payroll (other than that of the key employees), heat, light, power, and expenses of maintaining properties not destroyed (such as taxes, depreciation, and maintenance thereof). These items can be picked up by analyzing the running expense accounts. In no event does the policy pay expenses required to be insured unless it is proved that they continue after the fire.

Coinsurance clauses are also applicable in use and occupancy insurance.

A simple procedure to arrive at use and occupancy value for the past twelve months is as follows:

Total Sales		....
Deduct:		
Cost of Merchandise (Opening Inventory + Purchases − Closing Inventory)	....	
Ordinary Labor Payroll	....	
Light, Heat, and Power	....	
Total deductions		....
Actual 100% use and occupancy value for the period		....

The foregoing procedure is predicated on the assumption that all expenses other than ordinary payroll and light, heat, and power will continue at the same cost as if the business were operating.

A more exact method is one considered in the light of a problem in arithmetic or algebra, as follows:

Let x = Use and Occupancy Insurable Interest Each Day
a = Expenses That Do Not Continue During Suspension of Business
b = Selling Price of Merchandise
c = Cost of Merchandise
d = Number of Working Days in the Month

Then:

$$\frac{b - c - a}{d} = x$$

Example

The expenses of a business for a given month were determined as follows:

Item	*Total*	*Part of Expense That Must Continue During Suspension*	*Part of Expense That Will Not Continue During Suspension*
Payroll	$45,000		
Salaries and Wages of Key Employees Who Must Be Retained.		$20,000	
Salaries and Wages of Employees Not Retained			$25,000
Power	750	—	750
Heat and Light	525	225	300
Leasehold . .	1,200	1,200	—
Advertising	1,725	725	1,000
Taxes	950	950	---
Insurance .	1,375	500	875
Interest	525	525	—
Other Expenses	1,950	875	1,075
	$54,000	$25,000	$29,000

Find the estimated amount of insurance to be carried for each day of the month if sales are estimated to be $181,500 and cost of merchandise sold $109,500. Average number of working days each month is 25.

Solution

$$\frac{b - c - a}{d} = x \qquad \frac{181{,}500 - 109{,}500 - 29{,}000}{25} = 1{,}720$$

Therefore, on the basis of estimates, $1,720 is the amount of insurance to be carried for each day in the month.

The same result may be obtained in the following manner, using the estimates given:

Sales for the Month	$181,500
Less: Cost of Sales...........	109,500
Gross Profit............	72,000
Total Expenses.............	54,000
Net Profit.................	18,000
Add: Expenses That Must Continue During Suspension	25,000
Use and Occupancy Value for the Month .	$ 43,000

43,000 ÷ 25 = 1,720

Problems

1. The gross profit of a business was $200,000 after charging raw materials and payroll into manufacturing cost, but excluding light, heat, and power. Continuing payroll of key men was fixed at $20,000. If the policy contained the 80% clause, what was the required amount of insurance?

2. An audit of the expense accounts of the company insured in Problem 1 showed that items that would not have to be continued after the fire totaled $60,000. What amount would be collectible for a twelve-month period, other facts being as stated in Problem 1?

3. Assume that it takes 15 months to rebuild the plant. How much insurance would be collectible?

4. If a manufacturer on a Sept. 30 fiscal-year basis had a fire on April 1 and it is shown by previous experience that the following six months are the most profitable—in fact, that $66\frac{2}{3}\%$ of the net earnings are made in that period—would the adjustment take this into consideration, or would it be made on an average for the year?

5. If the conditions in Problem 4 were reversed, what earnings would the adjustment reflect?

6. Compute the use and occupancy value from the following data: Beginning inventory, $170,482.66; ending inventory, $171,721.77, manufacturing—including raw materials, labor, light, heat, and power, maintenance, depreciation, administration, insurance, taxes, interest, advertising, and all other expenses, $3,409,658.42. Fixed charges that are included in the foregoing and that are expected to continue are: administrative salaries, $35,200; interest, $4,800; taxes, $9,961.22; dues and pledges, $6,150; credit information, $235, insurance, $7,918.49; salaries of office, supervisors, and foremen that will have to be retained, $237,075; miscellaneous expenses, $42,395.62. Sales were $3,551,708.81.

Group life insurance. While this type of insurance is a part of the subject of life insurance, it is presented in this chapter because it is a common form of business insurance. The principles of life insurance are presented in another chapter.

Group life insurance affords employees of a business with ordinary life insurance at low cost so long as they are employed by the particular employer, as the employer pays a part of the premium. The operation of this type of insurance is best explained by an example of an actual plan.

Group Life Plan

1. *Eligibility.* The following plan of group life insurance is offered to all present employees of the company who will have completed six months or more of continuous service on November 11, 19—, and to all new employees after they have been with the company for six months.

2. *Amounts of insurance.* The amount of insurance available to each employee under age 65, nearest birthday, will be based on annual earnings as follows:

Class	*Annual Earnings*	*Life Insurance*
1.	Less than $1,200	$1,000
2.	$1,200 but less than $2,200	1,500
3.	$2,200, but less than $2,800	2,000
4.	$2,800, but less than $3,200	2,500
5.	$3,200, but less than $3,800	3,000
6.	$3,800, but less than $4,200	3,500
7.	$4,200, but less than $4,800	4,000
8.	$4,800, but less than $5,200	4,500
9.	$5,200 and over	5,000

3. *Cost of insurance.* The monthly cost of the insurance will be based on the employee's insurance age on each anniversary date of the plan, as shown in the following schedule:

Attained Age on Policy Anniversary Each Year	*Employee's Monthly Contribution per $1,000 of Insurance*
Age 44 and under	$0.70
Ages 45 to 54, inclusive	1.00
Ages 55 to 59, inclusive	1.50
Age 60 and over	1.80

Problems

1. Employee Y is 42 years of age and his earning classification is Class 5. What is the monthly deduction for his insurance?

2. If Y were 14 years older, what would be the monthly deduction?

3. B is 46 years of age and earns $3,000 a year. How much insurance is available to him, and what will be his monthly contribution?

4. Company X insures each of its employees for $1,000. Under age 50 the cost to the employee is 60 cents a month; at age 50 or over, the cost is $1.00 a month. There are 54 employees, classified as follows:

Age	*Number*
18	1
22	6
25	10
29	4
30	7
45	12
47	8
52	2
56	3
58	1

What is the amount of the monthly payroll deduction?

5. The manual shows the cost of group insurance on a monthly basis to be as follows:

Age	*Premium*
18	$ 51
22	53
25	54
29	.55
30	.55
45	.80
47	.90
52	1 26
56	1.71
58	2.00

With the number in each age group being that given in Problem 4, what is the amount of insurance premium that is borne by Company X?

Health insurance. Some plans are contributory and others non-contributory. In either case, the benefits are much the same; but in contributory plans the employee pays a part of the cost in the form of a monthly premium deducted from wages, while in the non-contributory plans the cost is borne by the employer. Few businesses have their own insurance departments, most of the plans being handled by insurance companies under a group plan.

Incapacities include sickness and non-occupational accidents (occupational accidents being covered by Workmen's Compensation Insurance), but the employer usually reserves the right to withhold benefits if the incapacity is the result of the employee's misconduct or negligence.

The following examples are illustrative of the many ways in which the factor of service is employed to favor the veteran worker.

Example 1

Length of Service	*Amount and Duration of Disability Benefits*
Under 2 years	Such practice as the company may establish
2 but less than 5 years	Full pay 4 weeks, half-pay 9 weeks
5 but less than 10 years	Full pay 13 weeks, half-pay 13 weeks
10 years and over	Full pay 13 weeks, half-pay 39 weeks

Example 2

Length of Service	*Amount and Duration of Disability Benefits*
1 but less than 10 years	50% of wages
10 but less than 30 years	75% of wages
30 years and over	100% of wages
	Maximum: 26 weeks in 3 years

Example 3

Length of Service	*Amount and Duration of Disability Benefits*
6 months but less than 1 year	35% of wages; maximum: $14.00 per week, for 6 weeks
1 but less than 2 years	50% of wages; maximum: $20.00 per week, for 13 weeks
2 but less than 3 years	60% of wages; maximum: $24.00 per week, for 13 weeks
3 but less than 4 years	70% of wages; maximum: $28.00 per week, for 26 weeks
4 years and over	75% of wages; maximum: $30.00 per week for 26 weeks.

Problems

1. *A* was insured under the plan in Example 1. He was employed for 3 years and became incapacitated for a period of 6 weeks. His average weekly wage was $35.80. What amount of disability benefit was he entitled to receive?

2. *B* was insured under the plan in Example 2. He had been with the same employer for 12 years. Two years ago he drew compensation for 8 weeks, and last year for 12 weeks. This year he was again incapacitated for a period of 8 weeks. If his average weekly wage was $45.00, what amount of disability benefit was he entitled to this year?

3. *C* was employed by an employer using the plan in Example 3, and had worked for this employer for a period of 6 years. He became incapacitated when receiving a weekly salary of $60.00, and was unemployed for 10 weeks. What was the amount of compensation paid?

Workmen's compensation insurance. This type of insurance is financial protection against loss of time for the wage-earning group, resulting from accident and occupational sickness while on duty. The cost is levied on the employer in the form of a premium on the payroll classified according to the hazard of occupation. A few states have their own Workmen's Compensation Insurance Departments, but in most states the insurance is carried by the insurance companies specializing in this type of insurance, generally referred to as *casualty insurance companies.*

Problems

Find the cost of workmen's compensation insurance on payrolls divided into four classifications with respective rates as follows:

1. $239,530.39 @ .611 per C
75,535.62 @ .519 per C
241,327.85 @ .081 per C
99,791.48 @ .586 per C

2. $272,584.07 @ .611 per C
91,856.68 @ .519 per C
292,258.87 @ .081 per C
148,735.42 @ .586 per C

3. $254,248.83 @ .581 per C
79,950.31 @ .548 per C
272,368.08 @ .085 per C
105,553.36 @ .564 per C

4. A deposit of $100.00 was made on a public liability policy. The payroll audit was as follows:

$381,839.77 @ .052 per C
294,212.15 @ .026 per C
138,631.05 @ .026 per C

What amount of additional premium was due on completion of the payroll audit?

*

CHAPTER 11

Payroll Records and Procedure

Requirements. The term *payroll records* has gained new significance since enactment of the Social Security Act and more recently the Current Tax Payment Act of 1943. Formerly each business handled its payroll system in accordance with its own particular needs. Now payroll systems are becoming more or less standardized in so far as certain information must be provided in order to meet the tax requirements.

Requirements at the time of wage payments are that the employer must deduct the taxes, both Federal Old Age Benefit Tax and Withholding Tax. Tax legislation has not dictated the form of records to be kept, but regulations have stipulated that certain information must be available, and that a statement shall be furnished the employee on or before January 31 of the succeeding calendar year, or, on the day on which the last payment of wages is made where employment is terminated before the close of the calendar year. Records needed are best determined from the reports required.

For operational purposes many employers give a pay statement with each wage payment. The pay detail can be shown on a stub attached to the pay check, on a duplicate of the pay check, on the cash pay envelope, or on a separate slip.

Payroll procedure. Payroll procedure involves, first of all, the production of the time card. which the individual employee either fills out or else inserts in the time recorder at stated times and which therefore contains the basic information for other records.

The following forms contain information transferred from the time card: the payroll summary sheet, the pay check or the pay envelope (if a pay envelope is used, a pay receipt is also required), and the individual employee's historical earning record.

Timebooks. Another method that is still employed to quite an extent entails the work of timekeepers who keep time books. The pencil or pen records that these timekeepers turn in show that individual employees work a certain number of hours and fractions of hours on particular work.

Time-clock cards. Time-clock cards provide a written record of the time each employee is on duty. The employee's time card

may be either a job ticket covering a single job, or a daily time report recording all jobs worked upon during the day. Attendance records or the In-and-Out clock cards are the controls on total time. A time-clock card must be prepared for each individual employee for each day or each pay period, according to the system in use. Each card must be imprinted with the day or the pay period date, clock number, employee's name, and possibly other identifying data. It shows the date, identification of the job and the employee, starting and stopping time, rate, elapsed time, and amount earned when completed. Piecework records also show the number of pieces produced. At the end of the day the elapsed time is computed from the clock registration and checked with that shown on the attendance records.

No. 138 PAY ENDING 8/28/46
NAME Ed Wolper-3

DEDUCTIONS
FOAB .43
INC TAX 2.00
War Bonds 4.34
Hosp. .80
Ins. .75
TOTAL $ 8.32

1	7 55	7 59	7 52	7 49				
2					8 03			
3								
4		11 30						
5	12 01		12 03	12 02	12 00			
6	12 28	12 24	12 29	12 27	12 26			
7								
8								
9								
10	4 32	4 35	4 32	4 38	4 33			
11								
12								
13								
14								
15								
16								
DAILY TOTAL	8	7½	8	8	8			

HRS 39½ @ 1.10 AMT. 43.45 TOTAL PA[Y]
TOTAL DEDUCTIO[NS]
TOTAL PAY 43.45 NET PAY

DAILY COST CARD

No. 138 RATE 1.10
NAME Ed Wolper

TIME IMPRINTS	TIME USED		QUANTITY	JOB NO	COST
		F			
		S			
AUG 23 16 4	4	F	2	275	44
AUG 23 16 0		S			
AUG 23 16 0	9	F	4	431	99
AUG 23 15 1		S			
AUG 23 14 9	9	F	6	701	99
AUG 23 14 0		S			
AUG. 23 13 9	13	F	8	598	143
AUG 23 12 6		S			
AUG 23 12 0	7	F	4	611	77
AUG 23 11 3		S			
AUG 23 11 2	12	F	10	722	132
AUG. 23 10 0		S			
AUG 23 10 0	4	F	3	490	44
AUG 23 9 6		S			
AUG 23 9 5	7	F	15	266	77
AUG 23 8 8		S			
AUG 23 8 7	7	F	12	937	77
AUG 23 8 0		S			
TOTALS	72				792

DATE 8/23 FOREMAN O K. C. E. Hammond

In and Out Clock Card and Daily Cost Card

The illustration is that of the weekly In-and-Out clock card from which the payroll is prepared and the daily cost card used for cost accounting purposes. The In-and-Out clock card illustrated shows the exact minute of entering and leaving. Some clocks register this time in tenths of an hour instead of the exact time. The daily cost card shows hours and tenths, beginning at the bottom and reading toward the top, so arranged to facilitate computation of elapsed time. The closing hour, 16.4, is 24 minutes past 4 o'clock.

The In-and-Out card shows 8 hours' elapsed time on Monday, the factory hours being from 8 A. M. to 12 M., and 12:30 P. M. to 4:30 P. M. Of course, it is impossible to check in and out on the specified hour, and a certain tolerance is allowed. Different companies have varying rules regarding tardy registrations. Wage and Hour inspectors object to too early registration, and more than 15 or 20 minutes early is likely to be counted as overtime.

The daily cost card shows 7.2 hours' productive time; therefore, the difference of .8 of an hour is nonproductive time. A reconciliation can be made showing where the .8 of an hour was not utilized on productive work.

.1	Between Jobs 837 and 266
.1	Between Jobs 266 and 490
.1	Between Jobs 722 and 611
.1	At noon (Starting time being 12.6 instead of 12.5.)
.1	Between Jobs 598 and 701
.2	Between Jobs 701 and 431
.1	At close of day (Finishing time being 16.4 instead of 16.5.)
.8	Total lost or nonproductive time

Deductions. Fixed or standard deductions, such as group insurance, employees' benefit association dues, hospital service dues, and so forth, can be entered at the time the card is made up. Federal Old Age Benefit Tax and Withholding Tax cannot be entered until earnings are computed.

At the present time, F. O. A. B. Tax is 1% of earnings. A portion of the Withholding Tax schedule effective January 1, 1946, that for a weekly payroll, is presented for use with the problems.

Withholding exemptions. For income tax computations, the personal exemption is on a per capita basis; therefore, withholding exemptions are on a per capita basis as follows:

A single person is allowed one exemption.

Husband and wife have two exemptions: if both are working, either spouse may take both exemptions or each may take one; if one is not working, the other may take both exemptions.

One exemption may be taken for each dependent (a person whose income is less than $600 a year, who is closely related to the taxpayer, and for whom the taxpayer provides more than one-half the support).

The number of exemptions claimed determines the proper column to be used in the wage-bracket tables in determining the tax to be withheld.

Employees' names on time cards and payrolls are marked to indicate the number of withholding exemptions claimed. Find the employees' earnings in the two columns at the left; where this line intersects the exemption column, the amount of tax to be withheld is shown.

If the payroll period with respect to an employee is WEEKLY

And the wages are—		And the number of withholding exemptions claimed is—										
At least	But less than	0	1	2	3	4	5	6	7	8	9	10 or more
		The amount of tax to be withheld shall be—										
$0	$11	17% of wages	$0	$0	$0	$0	$0	$0	$0	$0	$0	$0
$11	$12	$2.00	.10	0	0	0	0	0	0	0	0	0
$12	$13	2.10	.30	0	0	0	0	0	0	0	0	0
$13	$14	2.30	.50	0	0	0	0	0	0	0	0	0
$14	$15	2.50	.70	0	0	0	0	0	0	0	0	0
$15	$16	2.70	.80	0	0	0	0	0	0	0	0	0
$16	$17	2.80	1.00	0	0	0	0	0	0	0	0	0
$17	$18	3.00	1.20	0	0	0	0	0	0	0	0	0
$18	$19	3.20	1.30	0	0	0	0	0	0	0	0	0
$19	$20	3.30	1.50	0	0	0	0	0	0	0	0	0
$20	$21	3.50	1.70	0	0	0	0	0	0	0	0	0
$21	$22	3.70	1.80	0	0	0	0	0	0	0	0	0
$22	$23	3.80	2.00	.20	0	0	0	0	0	0	0	0
$23	$24	4.00	2.20	.40	0	0	0	0	0	0	0	0
$24	$25	4.20	2.40	.50	0	0	0	0	0	0	0	0
$25	$26	4.40	2.50	.70	0	0	0	0	0	0	0	0
$26	$27	4.50	2.70	.90	0	0	0	0	0	0	0	0
$27	$28	4.70	2.90	1.00	0	0	0	0	0	0	0	0
$28	$29	4.90	3.00	1.20	0	0	0	0	0	0	0	0
$29	$30	5.00	3.20	1.40	0	0	0	0	0	0	0	0
$30	$31	5.20	3.40	1.60	0	0	0	0	0	0	0	0
$31	$32	5.40	3.60	1.70	0	0	0	0	0	0	0	0
$32	$33	5.60	3.70	1.90	.10	0	0	0	0	0	0	0
$33	$34	5.70	3.90	2.10	.20	0	0	0	0	0	0	0
$34	$35	5.90	4.10	2.20	.40	0	0	0	0	0	0	0
$35	$36	6.10	4.20	2.40	.60	0	0	0	0	0	0	0
$36	$37	6.20	4.40	2.60	.80	0	0	0	0	0	0	0
$37	$38	6.40	4.60	2.80	.90	0	0	0	0	0	0	0
$38	$39	6.60	4.80	2.90	1.10	0	0	0	0	0	0	0
$39	$40	6.80	4.90	3.10	1.30	0	0	0	0	0	0	0
$40	$41	6.90	5.10	3.30	1.40	0	0	0	0	0	0	0
$41	$42	7.10	5.30	3.40	1.60	0	0	0	0	0	0	0
$42	$43	7.30	5.40	3.60	1.80	0	0	0	0	0	0	0
$43	$44	7.40	5.60	3.80	2.00	.10	0	0	0	0	0	0
$44	$45	7.60	5.80	4.00	2.10	.30	0	0	0	0	0	0
$45	$46	7.80	6.00	4.10	2.30	.50	0	0	0	0	0	0
$46	$47	8.00	6.10	4.30	2.50	.60	0	0	0	0	0	0
$47	$48	8.20	6.30	4.50	2.60	.80	0	0	0	0	0	0
$48	$49	8.40	6.50	4.60	2.80	1.00	0	0	0	0	0	0
$49	$50	8.60	6.60	4.80	3.00	1.20	0	0	0	0	0	0
$50	$51	8.80	6.80	5.00	3.20	1.30	0	0	0	0	0	0
$51	$52	8.90	7.00	5.20	3.30	1.50	0	0	0	0	0	0
$52	$53	9.10	7.20	5.30	3.50	1.70	0	0	0	0	0	0
$53	$54	9.30	7.30	5.50	3.70	1.80	0	0	0	0	0	0
$54	$55	9.50	7.50	5.70	3.80	2.00	.20	0	0	0	0	0
$55	$56	9.70	7.70	5.80	4.00	2.20	.40	0	0	0	0	0
$56	$57	9.90	7.90	6.00	4.20	2.40	.50	0	0	0	0	0
$57	$58	10.10	8.10	6.20	4.40	2.50	.70	0	0	0	0	0
$58	$59	10.30	8.30	6.30	4.50	2.70	.90	0	0	0	0	0
$59	$60	10.50	8.40	6.50	4.70	2.90	1.00	0	0	0	0	0
$60	$62	10.70	8.70	6.80	5.00	3.10	1.30	0	0	0	0	0
$62	$64	11.10	9.10	7.10	5.30	3.50	1.60	0	0	0	0	0
$64	$66	11.50	9.50	7.50	5.60	3.80	2.00	.20	0	0	0	0
$66	$68	11.90	9.90	7.80	6.00	4.10	2.30	.50	0	0	0	0
$68	$70	12.20	10.20	8.20	6.30	4.50	2.70	.80	0	0	0	0
$70	$72	12.60	10.60	8.60	6.70	4.80	3.00	1.20	0	0	0	0
$72	$74	13.00	11.00	9.00	7.00	5.20	3.30	1.50	0	0	0	0
$74	$76	13.40	11.40	9.40	7.30	5.50	3.70	1.90	0	0	0	0
$76	$78	13.70	11.70	9.70	7.70	5.90	4.00	2.20	.40	0	0	0
$78	$80	14.10	12.10	10.10	8.10	6.20	4.40	2.50	.70	0	0	0
$80	$82	14.50	12.50	10.50	8.50	6.50	4.70	2.90	1.10	0	0	0
$82	$84	14.90	12.90	10.90	8.80	6.90	5.10	3.20	1.40	0	0	0
$84	$86	15.30	13.20	11.20	9.20	7.20	5.40	3.60	1.70	0	0	0
$86	$88	15.60	13.60	11.60	9.60	7.60	5.70	3.90	2.10	.30	0	0
$88	$90	16.00	14.00	12.00	10.00	8.00	6.10	4.30	2.40	.60	0	0
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
$100	$105	18.50	16.50	14.50	12.50	10.50	8.50	6.60	4.70	2.90	1.10	0
$105	$110	19.50	17.50	15.50	13.50	11.40	9.40	7.40	5.60	3.80	1.90	.10
$110	$115	20.40	18.40	16.40	14.40	12.40	10.40	8.40	6.40	4.60	2.80	1.00
$115	$130	21.40	19.40	17.30	15.30	13.30	11.30	9.30	7.30	5.50	3.70	1.80
$120	$125	22.30	20.30	18.30	16.30	14.30	12.30	10.30	8.20	6.30	4.50	2.70
[illegible]	[illegible]	23.20	21.20	19.20	17.20	15.20	13.20	11.20	9.20	7.20	5.40	3.50
[illegible]	[illegible]	24.20	22.20	20.20	18.20	16.20	14.10	12.10	10.10	8.10	6.20	4.40
[illegible]	[illegible]	25.10	23.10	21.10	19.10	17.10	15.10	13.10	11.10	9.10	7.10	5.20
[illegible]	$145	26.10	24.10	22.00	20.00	18.00	16.00	14.00	12.00	10.00	8.00	6.10
$145	$150	27.00	25.00	23.00	21.00	19.00	17.00	15.00	12.90	10.90	8.90	7.00
$150	$160	28.40	26.40	24.40	22.40	20.40	18.40	16.40	14.40	12.30	10.30	8.30
[illegible]	[illegible]	30.30	28.30	26.30	24.30	22.30	20.30	18.20	16.20	14.20	12.20	10.20
[illegible]	[illegible]	32.20	[illegible]	[illegible]	26.20	24.10	22.10	20.10	18.10	16.10	14.10	12.10
[illegible]	[illegible]	34.10	[illegible]	[illegible]	28.00	26.00	24.00	22.00	20.00	18.00	16.00	14.00
[illegible]	$200	35.90	[illegible]	[illegible]	29.90	27.90	25.90	23.90	21.90	19.90	17.90	15.80
		19 percent of the excess over $200 plus—										
$200 and over		36.90	34.90	32.90	30.90	28.80	26.80	24.80	22.80	20.80	18.80	16.80

Example

Brown's earnings are $29.00 for the week, and his number of withholding exemptions is 2. What is the amount of Withholding Tax?

Solution

In the columns at the left find the bracket $29.00 to $30.00, and follow across to the intersection of the column headed "2." The tax is found to be $1.40.

Problems

1-2. Complete the following weekly time cards. Notice that the daily recordings are made across the card instead of vertically, as in the illustration on page 108. Make the extension of elapsed time at the extreme right of the card.

1.

PAY ENDING 10/25

No. 24

NAME Garret Knight – 3

	IN	OUT	IN	OUT	IN	OUT	
	M 7 57	M 12 01	M 12 59	M 4 05			
	TU 8 01	TU 12 02	TU 12 58	TU 4 02			
	W 7 53	W 12 02	W 12 57	W 3 59			
	TH 7 59	TH 12 01	TH 1 01	TH 4 04			
	FR 7 56	FR 12 03	FR 12 56	FR 4 02			
	SA 7 58	SA 12 10					

DEDUCTIONS
F O A B. ______
WHD TAX ______
OTHER ______
TOTAL $ ______

HRS. ______ AMT ______
TOTAL PAY ______
TOTAL DEDUCTIONS ______
NET PAY $ ______

2.

No. 25

PAY ENDING Sept. 24

NAME Alice Wharton – 1

	IN	OUT	IN	OUT	IN	OUT	
	M 8 28	M 12 03	M 12 59	M 4 48			
	TU 8 25	TU 12 05	TU 12 57	TU 4 50			
	W 8 28	W 11 58	W 12 55	W 4 46			
	TH 8 32	TH 12 01	TH 12 55	TH 4 55			
	FR 8 17	FR 12 02	FR 1 01	FR 4 49			
	SA 8 30	SA 12 10					

DEDUCTIONS
F O A B ______
WHD TAX ______
INS ______ 16
TOTAL $ ______

HRS ______ 50¢ AMT ______
TOTAL PAY ______
TOTAL DEDUCTIONS ______
NET PAY $ ______

3-4. Complete the following semi-monthly time cards. A section of the Withholding Tax schedule for semi-monthly pay periods is given to enable you to compute the tax:

$54 to $56	$1.50
56 to 58	1 80
58 to 60	2 20

3.

No. 16 PAY ENDING Oct. 31

NAME Harry Burr - 2

DATE	IN	OUT	IN	OUT	IN	OUT	
16							
1/17	M 7 52	M 12 04	M 12 49	M 4 02			
2/18	TU 7 49	TU 12 02	TU 12 53	TU 4 03			
3/19	W 7 55	W 12 04	W 12 58	W 4 00			
4/20	TH 7 58	TH 12 03	TH 12 54	TH 4 04			
5/21	FR 8 00	FR 12 00	FR 12 55	FR 4 01			
6/22	SA 7 59	SA	SA 1 05	SA			
7/23							
8/24	M 7 57	M 12 02	M 1 00	M 4 06			
9/25	TU 7 45	TU 12 03	TU 12 59	TU 4 04			
10/26	W 7 54	W 12 00	W 12 57	W 4 06			
11/27	TH 7 56	TH 12 04	TH 12 48	TH 4 02			
12/28	FR 7 48	FR 12 02	FR 12 56	FR 4 01			
13/29	SA 7 55		SA 1 13				
14/30							
15/31	M 7 50	M 12 01	M 12 57	M 4 03	M 4 29	M 9 34	

DEDUCTIONS

F O A B ______

WHD TAX ______

OTHER ______

TOTAL $ ______

HRS ______ @ 60 ______ AMT ______

______ ______ ______

TOTAL PAY ______

TOTAL DEDUCTIONS ______

NET PAY $ ______

The following problem contains overtime, which is to be computed at time and one-half. At the bottom of the time card, enter the regular hours on the first line and the overtime hours on the second line. You will notice that the rate has already been adjusted to one and one-half times the regular rate.

4.

No. 14 PAY ENDING Sept. 30

NAME Richard Knight – 2

DATE	IN	OUT	IN	OUT	IN	OUT	
16	FR 7 50	FR 12 05	FR 12 30	FR 4 35			
1/17							
2/18							
3/19	M 7 58	M 12 02	M 12 29	M 4 33			
4/20	TU 7 57	TU 12 00	TU 12 29	TU 4 30	TU 4 59	TU 7 02	
5/21	W 7 49	W 12 03	W 12 28	W 4 38			
6/22	TH 8 13	TH 12 07	TH 12 29	TH 4 37			
7/23	FR 7 49	FR 12 01	FR 12 30	FR 4 39			
8/24							
9/25							
10/26	M 7 59	M 12 00	M 12 30	M 4 34			
11/27	TU 8 00	TU 12 02	TU 12 30	TU 4 30			
12/28	W 7 58	W 12 03	W 12 28	W 4 37			
13/29	TH 7 48	TH 12 05	TH 12 30	TH 4 38			
14/30	FR 7 50	FR 12 00	FR 12 32	FR 4 22			
15/31							

DEDUCTIONS ______	HRS ______ @ 65¢ AMT. ______
F O A B ______	______ 97½ ______
WHD TAX ______	TOTAL PAY ______
INS 21	TOTAL DEDUCTIONS ______
TOTAL $ ______	NET PAY $ ______

Payroll sheets. Whether payment is made by cash or check, listings of payments to employees are made on payroll sheets for record purposes, and the hours worked, rate of pay, gross earnings, deductions, and net pay are shown for each employee.

Piecework system. Quantity of work produced rather than number of hours worked is the basis of earnings under a piecework system. Under this system there is an incentive for the skilled worker to produce more, thereby increasing his earnings. The payroll is designed to record the number of pieces produced rather than the number of hours worked; hence, no provision need be made for overtime.

Problems

1. Complete the following section of a payroll sheet. Refer to the table of Withholding Tax for weekly payroll on page 110 to compute the Withholding Tax. F. O. A. B. is to be computed at 1% of gross earnings.

Dept.________ PAYROLL SHEET Date________

Name	Number Exemptions	Hours Worked: Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.	Total	Rate	Gross Pay	Deductions: Whd. Tax	F.O.A.B.	Ins.	Hosp.	Misc.	Total	Net Pay
John Horman	0	8	8	8	8	8			.65½				.75	.50	1.50		
F A Denecke	4	8	7	7¼	6¾	8½	4½		.72				.81	.50	2.25		
T P Engel	2	7½	7	8	8½	7½	4		.60				.68	.50			
R A Forbes	1	6	8	5½	8½	8½	4		.70				.86	.50	1.70		
M Jeakle	5	7½	7¾	7½	8	7	4		.55				.71	.50	.50		
H S Kraft	1	8	8	8	8	8			.65½				.77	.50	3.50		
J A Long	2	8	8	8	8	8			.65½				.73	.50			

2. Complete the following section of a payroll sheet. Refer to the table of Withholding Tax for weekly payroll on page 110 to compute the Withholding Tax. F. O. A. B. is to be computed at 1% of gross earnings.

Dept.________							PAYROLL SHEET								Date________	
	Hours Worked								Deductions							
Name / Number Exemptions	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.	Total	Rate	Gross Pay	Whd. Tax	F.O.A.B.	Ins.	Hosp.	Misc.	Total	Net Pay
M A Morgan 4	7½	6	7	8	8½	4	[illegible]	.85	34 85			.78	.50	2.50		
A B Noonan 3	6¾	7	7½	8	8	4½		.75				.82	.50	.50		
T P Smith 1	7¼	7¾	8½	8	8	4		1.10				.86	.50	1.00		
P A Stubel 2	7½	8	8	7¾	8½			.72½				.72	.50	.50		
J P Welker 2	7	8	7½	8½	8	4		.60				.84	.50			
T A Ziegler 2	8	8	8	8	8			.55				.71	.50	.50		
Ed Wolper *																

* Enter data from time card on page 108.

3. In the following problem, the regular earnings and overtime earnings are kept separate. Hours over 40 a week are overtime. Complete the payroll. Refer to the table of Withholding Tax for weekly payroll on page 110 to compute the Withholding Tax. F. O. A. B. is to be computed at 1% of gross earnings.

TIME SHEET
and
PAYROLL SUMMARY

For period ending ____________ 19___ Dept. No. 12

Employee's Name	Number Exemptions	Regis. No.	Hours Worked M	T	W	T	F	S	S	Total Hours	Rate	Regular Earnings	Overtime	Gross Pay	Deductions O.A.B.	Whd.	Ins.	Hosp.	Total	Net Pay	Check Number
R A Forbes	2	12-569	8	7	7^{3}	6^{7}	8^{5}	4^{5}		41^{9}	.62½	26 19	59	26 78	27		.75	.50			1459
M A Morgan	0	12-575	7	6	7	8	8	4			.85						.78	.50			1465
A B Noonan	1	12-576	6^{8}	7	7^{5}	8	8	4^{5}			.75						.82	.50			1466
T P Smith	4	12-577	7^{2}	7^{8}	8^{5}	8	8	4			.94						.86	.50			1467
P A Stubel	3	12-578	7^{5}	8	8	7^{8}	8^{5}				72½						.72	.50			1468
J P Welker	2	12-579	7	8	7^{5}	8^{5}	8	4			60						.84	.50			1469
T A Ziegler	2	12-580	8	8	8	8	8				.55						.71	.50			1470
Totals																					

4. Complete the following piecework payroll.

No.	Name	Number Exemptions	Oper. No.	Number Produced M	T	W	T	F	S	Total	Rate	Amt. Earned	Deductions S.S.	Inc. Tax	Ins.		Total Ded.	Net Pay	Ck. No.
87	John Oakes	2	256	21	20	24	23	22			.35				.50				
89	Wm Ames	4	321	15	18	17	18	16			.42				.65				
90	Art Cook	1	184	16	16	15	17	16			.39				.55				
91	Henry Lodge	1	275	24	23	24	26	25	—		.40				.60				
92	Roy Arth	2	260	22	21	24	23	24			.38				.65				
	Totals																		

Pay checks. After the payroll has been figured, the next operation is the writing of checks. Two types of payroll checks are illustrated, the stub portions on this page, and the check portions on the following page.

Form A

BOONE COMPANY,
BOONEVILLE, MO.

Statement of Employee's Earnings and Payroll Deductions

RATE	TIME WORKED			AMOUNT EARNED	OTHER COMPENSATION	TOTAL AMOUNT TAXABLE	DEDUCTIONS		
	Days	Hours	Standard Hours				Tax	Federal Old Age Tax	Miscel
1.00	5	40	40	40.00	4.00	44.00	2.10	.44	1.51

This is your statement of earnings and receipt for deductions as required by law. Save it carefully as it is the basis of any claim for Unemployment Insurance or Old Age Pension.

Form A is the type produced on a bookkeeping machine, writing from the time cards and making the payroll check, payroll summary sheet, and earnings record in a single operation.

Form B

Pay-Roll Remittance Voucher

Employee S. S. Acct No 341
Herman Shultz - 3
For Earnings to SEP 15 19
Hours Worked Reg 90 Overtime --

Amount Earned	90.00
Deductions.	
Fed. Old-Age Tax	.90
Inc Tax	3.70
Gr. Ins	1.00
Misc	5.30
Total Deductions.	10.90
Net Amount Paid	79.10

DETACH AND RETAIN THIS VOUCHER
Master Industries, Inc.

Form A (*Continued*)

BOONE COMPANY,
BOONEVILLE, MO.

PAYROLL CHECK

DATE	CHECK NO
AUG 19	56793

PAY TO THE ORDER OF

D.C. DANBURY 364 09 1898 EXACTLY $ 39.95

Not Good for Over Two Hundred Dollars

FIRST NATIONAL BANK
BOONEVILLE, MO.
89-11

Treasurer

This check not valid unless presented for payment within 60 days from date of issue.

Form B (*Continued*)

DATE SEP 15 19

Master Industries, Inc.
Lecenter, Minn.

No. 2317

Manufacturers National Bank
Lecenter, Minn.

PAY Seventy-nine and 10/100------------------------ $ 79.10

TO THE ORDER OF Herman Shultz - 3

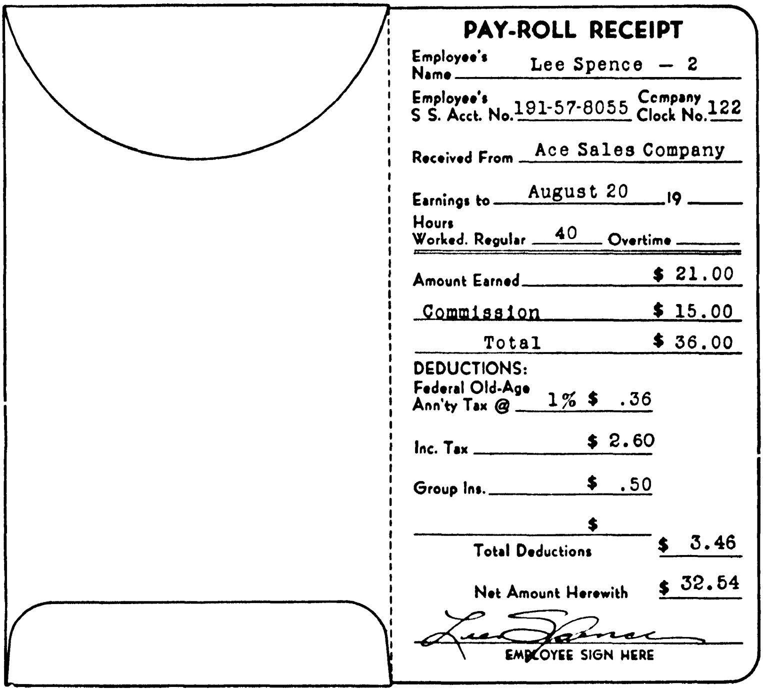
Master Industries, Inc.

PAY CHECK

BY

Form B is the type of check produced on a typewriter from data on the payroll summary sheet. Three operations are required: preparation of the payroll summary sheet, writing of checks, and, finally, posting to employees' earning records.

Pay envelopes and receipts. Some factories pay their employees by cash instead of by check. In such cases, pay envelopes and pay receipts are used.



PAY-ROLL RECEIPT

Employee's Name Lee Spence – 2

Employee's S. S. Acct. No. 191-57-8055 Company Clock No. 122

Received From Ace Sales Company

Earnings to August 20 19

Hours Worked. Regular 40 Overtime

Amount Earned	$ 21.00
Commission	$ 15.00
Total	$ 36.00

DEDUCTIONS:

Federal Old-Age Ann'ty Tax @ 1%	$ .36
Inc. Tax	$ 2.60
Group Ins.	$ .50
	$
Total Deductions	$ 3.46
Net Amount Herewith	$ 32.54

EMPLOYEE SIGN HERE

The flap, printed with the remittance data, when signed by the employee becomes a receipt for wages. The face of the envelope, printed the same as the flap, contains a carbon copy of the data and is the employee's permanent record.

Coin sheet and currency memorandum. Where employees are paid in cash, each employee receiving an envelope containing the exact amount of his net earnings, it is necessary to prepare a currency and coin sheet in order to have the correct number of units of each denomination.

Currency and Coin Sheet

No.	Name	Net Earnings		Currency 20	Currency 10	Currency 5	Currency 1	Coin 50	Coin 25	Coin 10	Coin 05	Coin 01
26	James Cook	43	82	2			3	1	1		1	2
28	Art Smith	57	76	2	1		1	1	1			1
30	Henry Brown	48	95	2		1	3	1	1	2		
31	Wm Jones	21	86	1			1	1	1	1		1
32	Ed. Adams	47	15	2		1	2			1	1	
	Totals	213	54	9	1	2	10	4	4	4	2	4

A currency memorandum, made up from the foregoing, is taken to the bank to enable the paying teller to make up the amount of money required in different denominations.

PAY ROLL

Date ______ 19__

FOR ______

Currency		Dollars	Cents
$100's			
50's			
20's	9	180	00
10's	1	10	00
5's	2	10	00
1's			
Silver			
Dollars	10	10	00
Halves	4	2	00
Quarters	4	1	00
Dimes	4		40
Nickels	2		10
Pennies	4		04
TOTAL		213	54

Problems

Rule currency and coin sheets and complete them for Problems 1, 2, and 3, pages 114, 115, and 116.

CHAPTER 12

Average

Simple average. The simple average of a group of items is determined by adding the items to be averaged and dividing the sum by the number of items added.

Example

From the following statistics, find the average rate per kilowatt hour for electrical energy:

New England States	2.88¢
South Atlantic States	2.77¢
Atlantic States	2.19¢
North Central States	1.88¢
Pacific Northwest	1.81¢

Solution

$$2.88 + 2.77 + 2.19 + 1.88 + 1.81 = 11.53$$
$$11.53 \div 5 = 2.306$$

Explanation. The number of items to be added is 5, and the sum is 11.53¢. 11.53 divided by 5 equals 2.306, therefore, 2.306¢ is the average rate per kilowatt hour.

Problems

1. The following delivery record shows the number of deliveries made each day by the five trucks of the delivery department:

TRUCKS

DAY	*No. 1*	*No. 2*	*No. 3*	*No. 4*	*No. 5*	TOTAL	AVERAGE
Monday	242	320	271	141	243		
Tuesday	217	328	393	182	218		
Wednesday	256	290	296	120	325		
Thursday	302	289	344	149	297		
Friday	293	306	301	216	218		
Saturday	317	365	423	227	303		
Total							
Average	271	316	338	[illegible]	[illegible]		

(*a*) Calculate the total number of deliveries made by each truck, and the daily averages for the week (vertical columns).

(*b*) Calculate the total number of deliveries made each day, and the average number of deliveries per truck (horizontal lines).

2. The monthly output of motor cars and trucks for one year was as follows:

January	231,728
February	323,796
March	413,327
April	410,104
May	425,783
June	396,796
July	392,076
August	461,298
September	415,285
October	397,096
November	256,936
December	233,135
Total	

What was the average monthly output for the year?

3. The sales of five clerks on a certain day were as follows:

A	$356.80
B	438.90
C	395.60
D	410.85
E	440.90
Total	

(*a*) Find the average sales.
(*b*) Which clerks sold above the average?
(*c*) Which clerks sold below the average?

Moving averages. Moving averages are a series of simple averages of statistics applicable to groups of an equal number of time units, each successive group excluding the data for the first time unit of the preceding group and including the data for the time unit immediately following those of the preceding group. For example, a yearly moving average, by months, may begin with an initial group including the data applicable to the twelve months of 1943. The next group would omit the data applicable to January, 1943, and include the data applicable to the remaining eleven months of 1943 and that applicable to the month of January, 1944.

Example

The labor costs in a certain manufacturing plant for the first six months of 1943 were as follows:

January	$3,363.17
February	3,644.15
March	4,472.90
April	3,209.20
May	3,415.40
June	4,152.05

The labor costs for the next two months were:

July	$3,824.06
August	4,015.25

What has been the average labor cost for each six months since January 1, 1943?

Solution

The labor cost for the period from January 1 to June 30 is the sum of the labor costs for each of the six months, or $22,256.87. The average for the period is $22,256.87 ÷ 6, or $3,709.48.

The average for the period from February 1 to July 31 is computed as follows:

Total: January 1 to June 30	$22,256.87
Deduct: January labor cost	3,363 17
	$18,893.70
Add: July labor cost	3,824.06
	$22,717 76

$22,717.76 ÷ 6 = $3,786.29

The average for the period from March 1 to August 31 is calculated in the same manner:

Total: February 1 to July 31	$22,717.76
Deduct: February labor cost	3,644 15
	$19,073 61
Add: August labor cost	4,015 25
	$23,088 86

$23,088.86 ÷ 6 = $3,848.14

Comparison of these averages, $3,709.48, $3,786.29, and $3,848.14, shows an increase for each period.

In permanent records, these averages should be tabulated.

1943	*Labor Cost*	*Moving Average*	*Increase or Decrease*†
January-June	$22,256 87	$3,709 48	$........
February-July	22,717 76	3,786 29	76.81
March-August	23,088.86	3,848.14	61.85

† Indicate decreases by means of daggers.

Further comparisons, based on the figures of prior periods, may be made in succeeding years. A column may be annexed to show the increase or decrease of the average of each six months' period compared with the simple average for the preceding year. Another column may be used to show the increase or decrease in the moving average for the current six months' period compared with the moving average for the same period of the preceding year.

Problems

1. Below are stated the labor costs, for the succeeding months, of the company in the preceding example; compute the moving averages.

September	$4,275.60
October	3,981 28
November	4,013 75
December	4,010 80

2. Using the averages obtained in Problem 1, complete the tabular record for the year 1943.

3. Find the simple average for the year 1943.

Progressive average. The method of progressive average is cumulative. The results of the latest period are added to the total previously computed, and the amount is divided by the previous divisor plus 1.

Example

Department A sales were: January, $5,364.00; February, $4,872.00; March, $5,024.00. Department B sales were: January, $2,561.00; February, $2,325.00; March, $2,753.00. Find the progressive monthly averages.

Solution

SALES RECORD

Dept.	*Jan.*	*Feb.*	*Total*	*Aver.*	*March*	*Total*	*Aver.*	*April*
A	5,364	4,872	10,236	5,118	5,024	15,260	5,087	
B	2,561	2,325	4,886	2,443	2,753	7,639	2,546	*etc.*

Explanation. Department A sales for January and February total $10,236.00. $10,236.00 ÷ 2 = $5,118.00, the average for the two months. $10,236.00 + $5,024.00 = $15,260.00, the total sales for the three months. $15,260.00 ÷ 3 = $5,087.00, the average for the three months. The record for the year would be completed in this manner.

The totals and averages of Department B are computed in the same way.

Problem

Using the above record and the following information, complete the record for the six months' period.

Department A sales:	
April	$5,986.00
May	6,125.00
June	6,398.00
Department B sales:	
April	2,482.00
May	2,593.00
June	2,715.00

Periodic average. Periodic average is simple average applied for several periods to statistics applicable to the same unit of time.

It may be used to show a variation in expenses, earnings, sales, and so forth.

Example

EXPENSES

Month	*1943*	*1942*	*1941*	*1940*	*Total*	*Average*
January........	$478 60	$392 85	$429 65	$356 00	$1,657.10	$414 28
February.......	462 37	529 83	531.33	535 35	2,058.88	514 72
March..........	347 92	629 89	432 45	567 89	1,978 15	494 54

Explanation. The expenses for January for the four years are totaled; the total, $1,657.10, divided by 4, the number of years shown, equals $414.28, the average monthly expense for January. The other averages are calculated in the same manner.

Problem

The output of a factory for the first quarter of the years 1943, 1942, 1941, and 1940 is shown in the following table:

Month	*1943*	*1942*	*1941*	*1940*	*Total*	*Average*
January	231,728	238,908	309,544	240,592	....	....
February.... .	323,796	304,735	364,180	283,577	....	....
March	413,327	394,513	434,470	374,425	....	....

Compute the periodic average.

Weighted average. Weighted averages take into account not merely the number of units to be averaged, but also the value of each unit.

The average-price method of pricing requisitions in cost accounting is illustrated in the following example.

Example

A stock record shows the following receipts:

4,800 lbs. @ 20¢
3,000 lbs. @ 18¢
4,000 lbs. @ 21¢

What is the average price per pound for the month?

Solution

4,800 lbs. @ 20¢	=	$ 960 00
3,000 lbs. @ 18¢	=	540.00
4,000 lbs. @ 21¢	=	840 00
11,800 lbs.	=	$2,340.00

2,340 ÷ 11,800 = 19.83, or 19.83¢ per pound, the average price.

Example

A manufactured product is composed of four ingredients, the relative proportions and costs per pound being as follows:

Material	*Pounds*	*Price per Pound*
A	1	$1 50
B	3	75
C	4	1 25
D	2	2 00

It was found in the second year that owing to price fluctuations, the raw material costs had increased as follows:

Material	*Per Cent*
A	50
B	100
C	10
D	25

What was the average per cent of increase in the cost of raw material composing the finished product?

Solution

Material	*Pounds*	*Cost per Lb.*	*Total Cost*	*Per Cent Price Increase*	*Increased Cost*
A	1	$1.50	$ 1 50	50	$ .75
B	3	.75	2 25	100	2.25
C	4	1.25	5.00	10	.50
D.	2	2.00	4 00	25	1 00
			$12.75		$4 50

4.50 ÷ 12.75 = 35.29%, the weighted average per cent

Problems

1. A stock card shows the following receipts:

3,000 lbs. @ 28¢
2,000 lbs. @ 27¢
1,500 lbs. @ 29¢
4,000 lbs. @ 30¢

What is the average price per pound?

2. X owns the following securities:

$3,000 Power Corp. 5½'s
$1,000 Alabama Company 6's
10 shares Northern Power Co.,
7% Preferred Stock, Par Value,
$100.00 a share
$2,000 Union Depot Co. 5's

What is the average rate of interest earned on X's investment, assuming that the securities were bought at par?

3. A product is manufactured from the following materials used in the relative oroportions given:

Material	*Pounds*	*Price per Lb.*
A	4	30¢
B	6	25¢
C	3	75¢
D	2	50¢

If the raw materials used advance in price at the following rates, what will be the average per cent of increase in the cost of the finished product?

A	25%	C	33⅓%
B	30%	D	50%

CHAPTER 13

Averaging Dates of Invoices

Definition. Averaging dates of invoices is the process of finding the date when several invoices due at different dates may be paid in one amount, without loss of interest to either debtor or creditor. This date is called the equated date of payment.

Use. The process of averaging the dates of invoices is most frequently used in bankruptcy settlements, where claims when filed with a trustee must show the average due date of the items if interest is to be obtained on overdue amounts. In general, the equated date is important in the settlement of bills of long standing, and in the fixing of the date of a note in settlement of invoices.

Term of credit. A term of credit is the time elapsing between the date of a bill and the date on which it becomes due; as, "Bill purchased January 10, Term of Credit 10 days." The due date would be January 20.

Average due date. The average due date is the date on which settlement of the complete account should be made by payment of the amount of the invoices, without charge for interest on overdue items or allowance for discount on prepaid items.

Focal date. The focal date is an assumed date of settlement with which the due dates of the several items may be compared, to determine the equated date of payment.

Any date may be used as the focal date, and the final result will be the same. In the interest method, any rate per cent may be used, and the result will be the same. However, 6% is usually used, as the computations are then less complicated.

In all calculations, use the nearest dollar. For example, for \$115.29, use \$115.00; and for \$161.84, use \$162.00.

When several bills are sold, some of which have a term of credit, first find the due date of those with a term of credit, and then find the equated date of the several bills.

With respect to bills with terms of credit, the due date of such bills, rather than the invoice date, is used in computing the equated date.

Do not use fractions of a day in determining the average date.

Methods. There are two methods in common use: the Product Method, and the Interest Method.

Rule for product method. Use as the focal date the last day of the month preceding the first item. Multiply each item by the number of days intervening between the assumed date and the due date of the item, and divide the sum of the several products by the sum of the account. Count forward from the assumed date the number of days obtained in the quotient. The result will be the average due date.

Example

Find the date at which the following bills of merchandise may be paid in one amount without loss to either party: Due January 1, $150.00; February 14, $200.00; April 20, $155.00; June 15, $200.00.

Solution by Product Method

(Focal date, Dec. 31)

Due January 1	$150	×	1	=	150
Due February 14	200	×	45	=	9,000
Due April 20	155	×	110	=	17,050
Due June 15	200	×	166	=	33,200
	705				59,400

59,400 ÷ 705 = 84 days.
84 days after December 31 is March 25.

Explanation. For convenience, assume December 31 as the date of settlement. On the first bill, which is due January 1, there would be interest for 1 day. On the second bill there would be interest from December 31 to February 14, or 45 days, which is equivalent to interest on $9,000.00 for 1 day. On the third bill there would be interest from December 31 to April 20, or 110 days, which is equivalent to interest on $17,050.00 for 1 day. On the fourth bill there would be interest from December 31 to June 15, or 166 days, which is equivalent to interest on $33,200.00 for 1 day.

If all the bills were paid December 31, the debtor would be entitled to interest on $59,400.00 for 1 day, or interest on $705.00, the amount of the account, for 84 days. It is evident that the bills could be paid at a time 84 days later than December 31, or March 25, without loss to either party.

Verification

The interest on $150.00 for 83 days is	$2.08
The interest on $200.00 for 39 days is	1.30
Total gain of interest to debtor	$3 38
The interest on $155.00 for 26 days is	$ .67
The interest on $200.00 for 82 days is	2.72
Total gain of interest to creditor	$3.39

The gain of interest to the debtor is on all bills paid after they are due.

The gain of interest to the creditor is on all bills paid before they are due. These two results should be equal, or within a few cents of the same amount. The reason for a little discrepancy is the fraction of a day which is disregarded in determining the due date of the account.

Solution by Interest Method

January 1	$150 00 for 1 day	=	$0 03, interest
February 14	200 00 for 45 days	=	1.50, interest
April 20	155 00 for 110 days	=	2 84, interest
June 15	200.00 for 166 days	=	5 53, interest
Total interest			$9.90

Interest on $705.00 for 1 day is $0.1175.
$9.90 ÷ $0.1175 = 84, or 84 days.

Explanation. Assume December 31, the last day of the month preceding the first item, to be the date of settlement. If the amount of the account, $705.00, is paid December 31, there will be a loss of interest to the debtor of $9.90. The interest on $705.00 for 1 day at 6% is $0.1175. It will take a principal of $705.00 as many days to produce $9.90 as the number of times that $0.1175 is contained in $9.90, or 84 days, the same result as was obtained by the product method.

Problems

1. Several invoices mature as follows:

April 12	$260 00	August 18	$120 00
May 25	500.00	September 2	300 00

At what date may the foregoing invoices be paid in one amount without loss to either party?

2. Calculate the average due date of the following invoices:

June 10	$400 00	August 15	$250 00
July 27	100 00	September 22	300.00

3. Calculate the average due date of the following invoices:

May 8	$275 00 on 60 days' credit
May 24	150 00 on 2 months' credit
June 10	300.00 on 90 days' credit
July 1	250 00 on 30 days' credit

CHAPTER 14

Equation of Accounts, or Compound Average

Definition. Equation of accounts, or compound average, is the process of finding the date when the balance of an account having both debits and credits can be paid without loss to either debtor or creditor. With respect to bills with terms of credit, the due date of the bill rather than the invoice date is used in computing the equated date. Credits other than for cash (such as non-interest-bearing notes) are extended to the due date thereof. Summarizing briefly, the date to be used for each debit and credit is the date when the item has a cash value of the amount shown in the entry.

Rule for the product method. After finding the date that each item has a cash value, use the last day of the month preceding the earliest date as the focal date for both sides of the account. Find the number of days between the focal date and the due date of each item; multiply each item by the number of days intervening between the focal date and the due date of the item. Find the sum of the products on both the debit and the credit sides of the account. Divide the difference between the sums of the debit and the credit products by the balance of the account. The quotient will be the number of days between the focal date and the average date of the account.

When to date forward or backward. The average date is forward from the focal date when the balance of the account and the excess of the products are on the same side (both debits or both credits); if they are on opposite sides, the average date is backward from the focal date.

Example

At what date may the balance of the following account be paid without loss of interest to either party?

Debits		*Credits*	
July 1, Mdse. 30 days......	$250.00	Aug. 15, Cash..............	$400.00
July 26, Mdse. 30 days......	425 00	Sept. 10, Cash	300.00
Aug. 15, Mdse. 60 days......	320.00	Sept. 20, Cash..............	150.00
Aug. 30, Mdse. 60 days......	500.00		

Solution by Product Method

Since the earliest date is July 31, the assumed focal date would be June 30. However, since July 31 is an end-of-month date, this date is used, as each multiplier is 31 less than it would be if June 30 were used.

July 31,	$ 250 00	× 0 =	00,000	Aug. 15,	$400 00	× 15 =	6,000
Aug. 25,	425 00	× 25 =	10,625	Sept. 10,	300 00	× 41 =	12,300
Oct. 14,	320 00	× 75 =	24,000	Sept. 20,	150.00	× 51 =	7,650
Oct. 29,	500 00	× 90 =	45,000				
	$1,495.00		79,625		$850 00		25,950

Debit side	$1,495 00	$79,625.00
Credit side	850.00	25,950 00
	$ 645.00	$53,675.00

$53,675.00 ÷ $645.00 = 83.

The equated date is, therefore, 83 days after July 31, or October 22.

Explanation. First find the due date of each item. For convenience, assume July 31, the earliest due date, as the day of settlement for all the items on each side of the account. Proceed as in the process of averaging dates, which was described in the preceding chapter. With July 31 used as the focal date, there is a loss of interest on the total debits equivalent to the interest on $79,625.00 for 1 day, and a gain of interest on the total credits equivalent to the interest on $25,950.00 for 1 day; or a net loss of interest equivalent to the interest on $53,675.00 for 1 day, which is equal to the interest on $645.00 for 83 days. It is evident that the date when there would be no loss of interest to either party must be 83 days after July 31, or October 22.

Solution by Interest Method

Debits

July 31,	0 days' interest on	$ 250 00 =	$ .00
Aug. 25,	25 days' interest on	425 00 =	1 77
Oct. 14,	75 days' interest on	320 00 =	4 00
Oct. 29,	90 days' interest on	500 00 =	7 50
		$1,495.00	$13.27

Credits

Aug. 15,	15 days' interest on	$ 400.00 =	$ 1 00
Sept. 10,	41 days' interest on	300 00 =	2 05
Sept. 20,	51 days' interest on	150 00 =	1 28
		$ 850.00	$ 4.33

Dr.	$1,495 00	Interest,	$13 27
Cr.	850 00	Interest,	4 33
6000)	645 00		$ 8.94

$ 1075 interest for one day.

$8.94 ÷ .1075 = 83.

The equated date of payment is, therefore, 83 days after July 31, or October 22.

Explanation. The explanation of the product method is applicable to the interest method. The variance is in finding interest on each item and dividing the interest balance by the interest for 1 day on the balance of the account.

Problems

Find the equated date in each of the following:

1.

Debits		*Credits*	
June 10, Mdse.	$500 00	July 5, Cash	$300 00
Aug. 20, Mdse.	100 00	Aug. 10, Cash	150 00
Oct. 30, Mdse.	250 00	Sept. 25, Cash	200 00

2.

Debits		*Credits*	
May 3, Mdse. 60 days	$300 00	June 20, Cash	$150.00
June 15, Mdse. 60 days	250 00	July 1, Note, 30 days without int.	200.00
July 20, Mdse. 30 days	175.00	Aug. 10, Note, 60 days, int., 6%	300.00
Aug. 27, Mdse. 60 days	225.00		

3.

Debits		*Credits*	
Mar. 1, Mdse. 30 days	$225.00	Mar. 31, Cash	$150.00
Mar. 20, Mdse. 2 mos	300 00	Apr. 15, Cash	100 00
Apr. 5, Mdse. 60 days	150.00	May 10, Cash	200.00

CHAPTER 15

Account Current

Definition. An account current is a transcript of the ledger account. It should show the dates on which sales were made, the term of credit for each item, cash payments, and, if settlements were made by note, the date and other details of each note.

Methods. Two methods are used in finding the amount due: the Interest Method, and the Product Method.

Example

Find the balance due January 1 on the following ledger account, which bears interest at 6%.

J. B. JOHNSON

Dr.		*Cr.*	
Sept. 1, Balance..........	$1,200 00	Oct. 1, Cash............	$1,000 00
Sept. 20, Mdse. 30 days ...	400 00	Nov. 10, Cash............	200.00
Oct. 30, Mdse. 30 days ...	520 00	Dec. 3, Cash............	400.00
Nov. 25, Mdse. 30 days....	350.00	Dec. 15, Note 10 days.....	300.00

Solution by Interest Method

J. B. JOHNSON

Dr.				*Cr.*			
Date Due	*Amount*	*Days*	*Interest*	*Date*	*Amount*	*Days*	*Interest*
Sept. 1	$1,200 00	122	$24 40	Oct. 1	$1,000 00	92	$15 33
Oct. 20	400 00	72	4.87	Nov. 10	200 00	52	1 73
Nov. 29	520 00	33	2.86	Dec. 3	400 00	29	1.93
Dec. 25	350 00	7	.41	Dec. 25	300 00	7	.35
	$2,470 00		$32.54		$1,900.00		$19.34
	1,900 00		19 34				
	$ 570 00	+	$13.20 = $583.20				

Explanation. The number of days opposite each entry is the actual number of days from the date of the item to January 1, the date which is taken as the focal date.

Solution by Product Method

Dr.						*Cr.*					
Date Due	*Amount*		*Days*		*Product*	*Date*	*Amount*		*Days*		*Product*
Sept. 1	$1,200	×	122	=	$146,400	Oct. 1	$1,000	×	92	=	$ 92,000
Oct. 20	400	×	73	=	29,200	Nov. 10	200	×	52	=	10,400
Nov. 29	520	×	33	=	17,161	Dec. 3	400	×	29	=	11,600
Dec. 25	350	×	7	=	2,450	Dec. 25	300	×	7	=	2,100
	$2,470				$195,210		$1,900				$116,100
	1,900				116,100						
	$ 570				$ 79,110						

The interest on $79,110 for 1 day = $ 13.20

$570.00 + $13.20 = $583.20

In some instances it is more convenient to find the equated due date, and then calculate the interest on the balance of the account from that date to the date of settlement.

Problems

1. Find the amount that will settle the following account Sept. 10, interest at 6%.

Dr.		*Cr.*	
Mar. 15, Mdse. 4 mos	$450 00	July 5, Cash.	$400 00
Mar. 30, Mdse. 60 days	375 00	July 30, Cash	375 00
Apr. 18, Mdse. 30 days	700 00	Aug. 15, Cash	690.00
May 15, Mdse. 4 mos.	620 00	Sept. 5, Cash	615 00
May 30, Mdse. 4 mos.	410 00		

2. Find the amount that will settle the following account on June 1, interest at 6%.

Dr.		*Cr.*	
Jan. 4, Mdse. 30 days.	$500 00	Feb. 20, Cash.	$300.00
Jan. 30, Mdse. 30 days.	200 00	Feb. 28, Note, 60 days with	
Feb. 5, Mdse. 30 days.	600 00	interest at 6% . . .	300 00
Mar. 1, Mdse. 30 days.	400.00	Mar. 20, Cash.	150 00

3. *A* borrowed $10,000.00 from a bank on January 2, giving a note secured by a mortgage for building a home, due in one year, with interest at 6%. From time to time the bank advanced him money to pay contractors' estimates. Before maturity the bank had actually advanced $9,000.00, as follows:

January 31.	$3,000 00
March 15	3,000 00
April 15	1,500.00
May 15.	1,500.00

On June 1, the following year, the maker of the note desires to pay it. (*a*) How should interest be computed? (*b*) What amount is due June 1?

CHAPTER 16

Storage

Definition. Storage is the charge made by a warehouse or depositary for the storing of goods until they are required for use or for transportation to some other point.

Running account. When goods are being received and delivered, the storage company keeps a running account, showing the dates at which goods are received and delivered, together with details of the number of packages, barrels, and so forth. Storage is charged for the average number of days for which one package, barrel, or box has remained in storage. The average number of days is divided by 30 to reduce the average number of days to months, or by 7 to reduce the average number of days to weeks, as the case may be; then the number of months or weeks is multiplied by the price per month or per week.

Example

The following is a memorandum of the quantity of salt stored with a storage company at 4¢ per barrel per term of 30 days' average storage.

Date	*Receipts*	*Deliveries*	*Balance*	*Time in Storage*	*Equivalent for 1 Day*
June 4	120 bbl.		120 bbl.	28 days	3,360 bbl.
July 2		20 bbl.	100 bbl.	18 days	1,800 bbl.
July 20	100 bbl.		200 bbl.	10 days	2,000 bbl.
July 30		50 bbl.	150 bbl.	11 days	1,650 bbl.
Aug. 10		100 bbl.	50 bbl.	15 days	750 bbl.
Aug. 25		50 bbl.	0 bbl.		
					9,560 bbl.

Explanation. 9,560 bbl. for 1 day are equivalent to 1 barrel for 9,560 days, and 9,560 divided by 30 (the number of days per term) equals $318\frac{2}{3}$ terms. In some cases a full month's storage is charged for any part of a month that goods remain in storage; in other cases, 15 days or less are called one-half of a month, and any period of over 15 days is counted as a whole month. $318\frac{2}{3}$ terms would be charged for as 319 terms, and $319 \times .04 = 12.76$. Therefore, $12.76 is the storage charge.

Problems

1. On the following memoranda, compute storage at 4¢ per barrel per term of 30 days' average storage:

Received		*Delivered*	
Feb. 10	300 bbl.	Feb. 20	150 bbl.
Feb. 19	150 bbl.	Mar. 5	200 bbl.
Mar. 12	500 bbl.	Mar. 15	400 bbl.
Mar. 30	300 bbl.	Apr. 14	300 bbl.

2. A grower stored 5,000 bushels of potatoes at $5\frac{1}{2}$¢ per cwt., the term being 30 days' average storage. The following is a memorandum of the transactions that occurred. Compute the amount of storage.

Received		*Delivered*	
Sept. 1	2,500 bushels	Nov. 4	500 bushels
Sept. 10	1,500 bushels	Dec. 10	600 bushels
Oct. 5	1,000 bushels	Jan. 15	750 bushels
		Feb. 1	1,500 bushels
		Mar. 18	750 bushels
		Apr. 2	900 bushels

CHAPTER 17

Inventories

Valuation of inventories. The bases of inventory valuation most commonly used by business concerns are: (*a*) cost; and (*b*) cost or market, whichever is lower. However, the average cost method is used in some instances—the tobacco industry, for example—and market value as a basis is used in grain and cotton inventories and in inventories of dealers in securities.

Cost or market, whichever is lower. In valuing inventories at cost or market, whichever is lower, a comparison of inventory totals at the two values is not sufficient. It is necessary to consider each item or group of similar items purchased at the same price, and to make the extension at the cost or market price, whichever is lower.

Example

One hundred tons of sugar (200,000 lbs.) were purchased at 7¢ a pound, and later, 50 tons (100,000 lbs.) were purchased at 6¢ a pound. The entire 150 tons were on hand at the close of the year, at which time the market value of sugar was $6\frac{1}{2}$¢ a pound. Compute the inventory at: (*a*) cost; and (*b*) at cost or market, whichever is lower.

Solution

(*a*)	200,000 lbs. @ .07	$14,000
	100,000 lbs. @ .06	6,000
	Inventory at cost	$20,000
(*b*)	200,000 lbs. @ .065	$13,000
	100,000 lbs. @ .06	6,000
	Inventory at cost or market, whichever is lower	$19,000

Problems

1.* Given the following inventory of a retail shop for children's clothing, toys, and so forth (correct as to quantities and values), state the amount which should be shown on a balance sheet as merchandise inventory, adopting the method of valuing inventory at cost or market, whichever is lower.

* C. P. A., Maryland.

Item	Value Per Unit Cost	Value Per Unit Market	Total Value Cost	Total Value Market
150 knit towels	$ 0 38	$ 0 35	$ 57.00	$ 52 50
16 crepe de chine carriage sets	10 00	12.50	160 00	200 00
125 lingerie and pongee hats	2.00	1.75	250 00	218.75
85 rubber bibs with sleeves	.50	.50	42 50	42.50
240 creepers	2 05	1 98	492 00	475 20
200 spring coats	9 50	10 00	1,900 00	2,000.00
50 spring coats	17 50	18 50	875 00	925 00
8 play yards	6 00	5.75	48 00	46.00
8 desks used in office	55.00	60 00	440 00	480 00
140 shirts	.75	.79	105 00	110.60
200 boys' wash suits	6 20	5 98	1,240 00	1,196 00
125 bloomers	1 85	1 80	231 25	225 00
5 cribs	21 00	19 98	105 00	99 90
12 electric trains	1 50	1.50	18 00	18 00
Total			$5,963 75	$6,089 45

2.* You are called in by the X. Y. Z. Clothing Company to advise them on the calculation of their inventory. They have always followed the policy of cost or market, whichever is lower. You are informed that the inventory will be used for the tax return, as well as for the annual report to stockholders.

Item	Value per Unit Cost	Value per Unit Market	Total Value Cost	Total Value Market
13 suits, grade A	$60 00	$55 00	$ 780 00	$ 715 00
12 suits, grade B	40 00	37 50	480 00	450 00
17 suits, grade C	30 00	30 00	510 00	510 00
7 suits, grade D	20.00	22 00	140 00	154 00
24 overcoats, grade 1	75 00	80 00	1,800 00	1,920 00
5 overcoats, grade 2	40 00	45 00	200 00	225 00
10 overcoats, grade 3	30 00	25 00	300 00	250 00
6 topcoats, grade X	20 00	17.50	120 00	105 00
9 topcoats, grade Y	15 00	12 50	135 00	112.50
18 topcoats, grade Z	10.00	11.00	180 00	198.00
Total			$4,645 00	$4,639 50

Which total figure would you advise the company to use for: (*a*) tax reports; (*b*) annual report to stockholders?

Average cost method. The general rule that the average cost method of valuing inventories will not be accepted for income tax purposes is subject to certain exceptions. In the tobacco industry, for example, tobacco is bought from the producer, usually in small quantities and at greatly varying prices. Different grades of tobacco are mixed and stored in hogsheads, and it is practically impossible to determine the exact cost of any particular hogshead. The inventory is therefore averaged monthly, according to grades, as follows:

First method. From the inventory of each grade at the beginning of the month is subtracted the amount of tobacco of that

* C. P. A., Wisconsin.

grade used, leaving so many pounds costing so many dollars; to this is added the tobacco of that grade purchased during the month, and a new average is determined. This is the inventory for the close of the month, and is consequently the opening inventory of the next month.

Example

STOCK CARD

Received				Issued			Balance		
Date	*Quantity*	*Rate*	*Amount*	*Date*	*Quantity*	*Rate*	*Quantity*	*Rate*	*Amount*
6–29	100,000	$1 00	$100,000				100,000	$1.00	$100,000
				9–1	80,000	$1 00	20,000	1 00	20,000
9–30	80,000	1.10	88,000				100,000	1 08	108,000
				12–5	30,000	1 08	70,000	1.08	75,600
12–10	125,000	.95	118,750				195,000	....	194,350
				12–18	20,000	1 08	175,000	....	172,750
Inv't 12–31	175,000	$0 987	$172,750						

Explanation. It will be noticed that the receipt of 125,000 at .95 on Dec. 10 was extended into the balance column in quantity and amount only, and that the issuance on Dec. 18 was made at the rate established on the first of the month. This is the method that is used when receipts are frequent, as it saves the time that would be required to compute a new rate after each receipt, and establishes a standard rate of issuance for the month.

Second method. When receipts are not frequent and are large in amount, a new average price is computed upon the entry of each receipt.

Example

STOCK CARD

Received				Issued			Balance		
Date	*Quantity*	*Rate*	*Amount*	*Date*	*Quantity*	*Rate*	*Quantity*	*Rate*	*Amount*
6–29	100,000	$1.00	$100,000				100,000	$1.00	$100,000
				9–1	80,000	$1.00	20,000	1 00	20,000
9–30	80,000	1.10	88,000				100,000	1.08	108,000
				12–5	30,000	1.08	70,000	1.08	75,600
12–10	125,000	.95	118,750				195,000	.99½	194,350
				12–18	20,000	99½	175,000	.99½	174,430
Inv't 12–31	175,000	$0.99½	$174,430						

Problems

1. Rule two stock cards as in the preceding example, and enter the following data. Compute the balances: (a) by the first method; and (b) by the second method.

	Received	*Issued*
July 5.....	80,000 units @ $0.90	Aug. 1....... 50,000 units
Aug. 15....	20,000 units @ 1.00	Dec. 2 ...... 20,000 units
Sept. 1.....	30,000 units @ 1.10	Dec. 20...... 30,000 units
Dec. 8.....	20,000 units @ 1.20	

2. Complete the following stock ledger card, using average-price method.

Stores Ledger

Actual Receipt Price	.15	.20			
Average Price	.15	.1708			

Name Bushings 3" Mall Part No. 56-678

Minimum 100 Maximum 300 Location L3

Drawing No. Unit

	REFERENCE			QUANTITY		ON HAND		
	Date	Number	Remarks	Received	Issued	Quantity	Value	
1	OCT 4	4523		100				1
2	OCT 10	34567			5			2
3	OCT 12	35654			10			3
4	OCT 13	38765			3			4
5	OCT 15	39458			12			5
6	OCT 16	4587		50				6
7	OCT 19	40156			10			7
8								8
9								9
10								10
11								11
12								12
13								13

"First-in, first-out" method of inventory. Where the same merchandise has been purchased at various prices during the year, and the goods on hand cannot be identified with specific invoices, the amount on hand at the end of the year may be inventoried at the latest purchase price. If, however, the quantity on hand is greater than the amount purchased at the last price, the balance may be inventoried at the next to the last purchase price, and so on. This method is termed "first-in, first-out method" of inventory.

Example

Inventory, December 31........................	275,000 units
Invoices:	
November 10....................	125,000 units @ $0.95 per C
September 5.....................	80,000 units @ 1.10 per C
June 10.........................	70,000 units @ 1.00 per C

How should the foregoing inventory be valued?

Solution

125,000 units @ \$0.95 per C	\$1,187.50
80,000 units @ 1.10 per C	880.00
70,000 units @ 1.00 per C	700.00
275,000 units inventoried	\$2,767.50

"Last-in, first-out" method of inventory. Under the "last-in, first-out" method inventories are valued at the cost of goods earliest acquired, and in computing profits from sales the cost of goods last acquired is used. This method will show smaller profits when prices are rising and larger profits when prices are falling than the "first-in, first-out" method. Businesses which use raw materials or other goods includable in inventory, which are subject to sharp price fluctuations; businesses in which the value of inventory is large compared with other assets and sales; and businesses in which production consumes an extended period are most likely to benefit from the use of this method. (Consult the Internal Revenue Code relative to the requirements incident to adoption and use of this method.)

Example

A has an opening inventory of 10 units at 10 cents a unit, and during the year he makes purchases of 10 units as follows:

January	1	@ .11	=	.11
April	2	@ .12	=	.24
July	3	@ .13	=	.39
October	4	@ .14	=	.56
	10			1.30

His closing inventory shows 15 units. What is the value of the closing inventory?

Solution

	10	@ .10	=	1.00
	1	@ .11 (Jan.)	=	.11
	2	@ .12 (Apr.)	=	.24
	2	@ .13 (July)	=	.26
Totals	15			1.61

Problem

Value the closing inventory, using the "last-in, first-out" method.

Opening inventory: 50 units at \$1.00
Production:
First quarter: 50 units at \$1.50
Second quarter: 100 units at \$1.75
Third quarter: 50 units at \$2.00
Fourth quarter: 100 units at \$2.25
Closing inventory: 150 units

Merchandise turnover. The number of times that the value of the inventory is contained in the cost of sales is the merchandise turnover.

The final inventory should not be used in computing turnover, unless it represents a normal inventory for the fiscal period, or is the first inventory that has been taken.

If a perpetual inventory system is in use, the monthly inventories should be added to the inventory at the beginning of the period, and the sum divided by the number of months in the fiscal period plus one. In a year there would thus be thirteen inventories—the one at January 1, and the twelve inventories at the ends of the months. When a perpetual inventory is not used, add the inventory at the beginning of the fiscal period to the one at the close of the period; then divide by two. The quotient will be the estimated average inventory for the period. If semiannual inventories are taken, use three inventories and divide by three. If quarterly inventories are taken, use five inventories and divide by five.

FORMULA

Cost of Sales ÷ Average Inventory at Cost = Rate of Turnover

Example

A department store found the average inventory of Department A for the fiscal period to be \$30,000. The cost of sales for the same period was \$120,000.

\$120,000 ÷ \$30,000 = 4, the rate of turnover.

An estimated inventory at the end of any period may be obtained by dividing the sales for the period by 100% plus the per cent of gross profit based on cost, and deducting the quotient from the total of purchases and first of period inventory. A more complete discussion of the gross profit test is given in Chapter 18.

Example

In the above example, assume that in Department A the total cost of merchandise was \$150,000, that the sales were \$144,000, and that the average profits were 20%. Using the per cent of gross profits to determine the average inventory, the solution would be as follows:

\$144,000 (sales) ÷ 120% = \$120,000, the cost of sales.
\$150,000 (total cost of goods) − \$120,000
(cost of sales) = \$30,000, the estimated inventory.
\$120,000 (cost of sales) ÷ \$30,000
(inventory) = 4, the rate of turnover.

Number of turnovers. The number of turnovers varies in different lines of business. Records show turnovers varying from 1 to more than 20, depending on the kind of business. It is possi-

ble to make a larger profit by several turnovers with a small mark-up* than by 1 or 2 turnovers with a large mark-up. Limited capital and frequent turnovers can produce a profit equal to that produced by a greater capital turned fewer times a year. If a merchant turns $1 eight times in the course of a year, he has used $\frac{1}{8}$ of the capital that would be required if the rate of turnover were 1.

Example 1

A merchant had a rate of mark-up of 50%, with a turnover of 1. He found that by using a rate of mark-up of 30% he had a turnover of 2. If his former sales were $300,000 annually, how much were his gross profits increased, provided he continued to use the same investment in merchandise?

$300,000 ÷ 150% = $200,000, cost of sales.
$300,000 − $200,000 = $100,000, gross profits.

Under the new policy he turns the $200,000 twice, the equivalent of $400,000 annually.

$400,000 at 30% = $120,000, profits.
$120,000 − $100,000 = $20,000, increased profits due to lowering the rate of mark-up and increasing the rate of turnover.

Example 2

What investment in merchandise would be required under the new policy to make the same amount of profits that was made under the old policy?

$100,000 ÷ 30% = $333,333.33, cost of goods sold to make profits of $100,000.

Since there were 2 turnovers, the cost of goods sold was twice the amount of the average inventory. Therefore:

$333,333.33 ÷ 2 = $166,666.67, the average inventory.

Hence, the merchant could make the same amount of gross profits with an investment $33,333.33 smaller than that required under his old policy.

Problems

1. A rate of mark-up of 30% results in 2 turnovers of an average inventory of $30,000. If the expenses of conducting the business are $8,000, what is the net profit?

2. A rate of mark-up of 20% results in 3 turnovers of an average inventory of $30,000. If expenses remain at $8,000, what is the net profit?

3. The cost of sales in Department B was $42,000. The average inventory was $12,000. What was the number of turnovers?

4. A merchant's sales amounted to $42.000. His average inventory was $10,000, and the average rate of mark-up was 40%. Find the number of turnovers.

* "Mark-up," as used in this text, refers to the addition made to the cost of merchandise to produce the selling price.

5. Commodity X, with a rate of mark-up of 40%, had a turnover of 2. With a rate of mark-up of 30%, it had a turnover of 3. If prior sales were \$56,000, find the sales and the increase in gross profit with the 30% rate of mark-up.

6. A rate of mark-up of 35% results in a turnover of 2 and in sales amounting to \$540,000. A rate of mark-up of 20% results in a turnover of 4. How much less capital under the latter plan is required to make as much profit as under the former plan?

7.* On January 1, a concern dealing in a single commodity had an inventory of merchandise which cost \$20,000. The goods were marked to sell at 125% of cost, and all subsequent purchases during the six months ending June 30 were marked at the same rate. The selling price of the inventory at June 30 was \$24,000. Purchases and sales by months were:

	Purchases (Cost)	*Sales (Selling Price)*
January	\$ 8,000	\$ 9,000
February	9,000	9,500
March	14,000	12,000
April	16,000	18,000
May	13,000	22,000
June	10,000	18,000

(*a*) Compute estimated inventories at cost price at the end of each of the six months.

(*b*) Compute the rate of turnover for the six months' period, using (1) the January 1 and June 30 inventories; (2) all the inventories.

(*c*) State which method gives the more accurate results.

Per cent of mark-down to net cost. If an item costs \$1 and is marked \$1.25, in order to sell the item for cost the price must be reduced 25¢. The marked price is the base when prices are reduced. 25¢ is $\frac{1}{5}$ of \$1.25. $\frac{1}{5} = 20\%$.

An item costs \$1 and is marked \$1.50. 50¢ reduction is $\frac{1}{3}$ of \$1.50, or $33\frac{1}{3}\%$.

Problems

Calculate the per cent of mark-down for each of the following items:

Item	*Cost*	*Marked Price*	*Per Cent of Mark-down to Produce Cost*
A	\$ 2 00	\$ 2 50	....
B	1 00	1.25	....
C	.35	.40	....
D	.80	1 00	....
E	15.00	25 00	....
F	3.50	4 00	....
G	.20	30	....
H	5.00	7 00	....
I	.08	.10	....
J	2.00	4 00	....
K	40 00	75 00	....
L	12.00	18.00	....

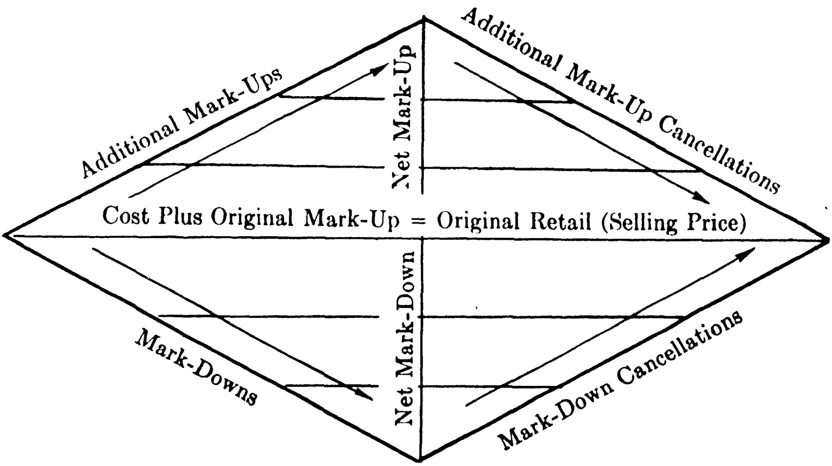
* American Institute Examination.

Computation of inventory by the retail method. The need for frequent inventories has led many department stores to adopt the "retail method" of computing inventories. The accuracy of the inventory by this method depends upon the care exercised in recording the mark-ups and the mark-downs of merchandise prices, and the classification of merchandise into departments and groups and sub-classes within the departments. In addition to the usual records showing sales (at selling price only), records are kept which show the opening inventory and purchases at cost and at retail (or selling) prices. An estimated inventory may be prepared from such records in the following manner.

INVENTORY COMPUTATION

	Cost	*Retail*
Inventory, beginning of period	\$ 6,000	\$ 8,000
Purchases during the period, including freight and cartage	74,000	111,200
Totals	\$80,000	\$119,200
(% Mark-on = \$39,000 ÷ \$119,200 or 32.8859%.)		
Sales		104,200
Inventory at retail		\$ 15,000
Estimated inventory = \$15,000 − (\$15,000 × 32.8859%) =		\$10,067.

The foregoing illustration does not take into consideration changes in selling price after the original mark-up. Price changes must be dealt with, and the retail mercantile business has terms for these changes that are not generally understood; therefore, to prevent any [illegible] the following diagram is presented and the terms explained.



Original mark-up. The amount by which the original retail price of an article exceeds the cost is the original mark-up.

Additional mark-up. An amount that increases the original retail price is an additional mark-up.

Additional mark-up cancellation. A reduction in the additional mark-up is an additional mark-up cancellation, and the amount cannot exceed the amount of the additional mark-up.

Net mark-up. The sum of additional mark-ups minus the sum of additional mark-up cancellations is the net mark-up.

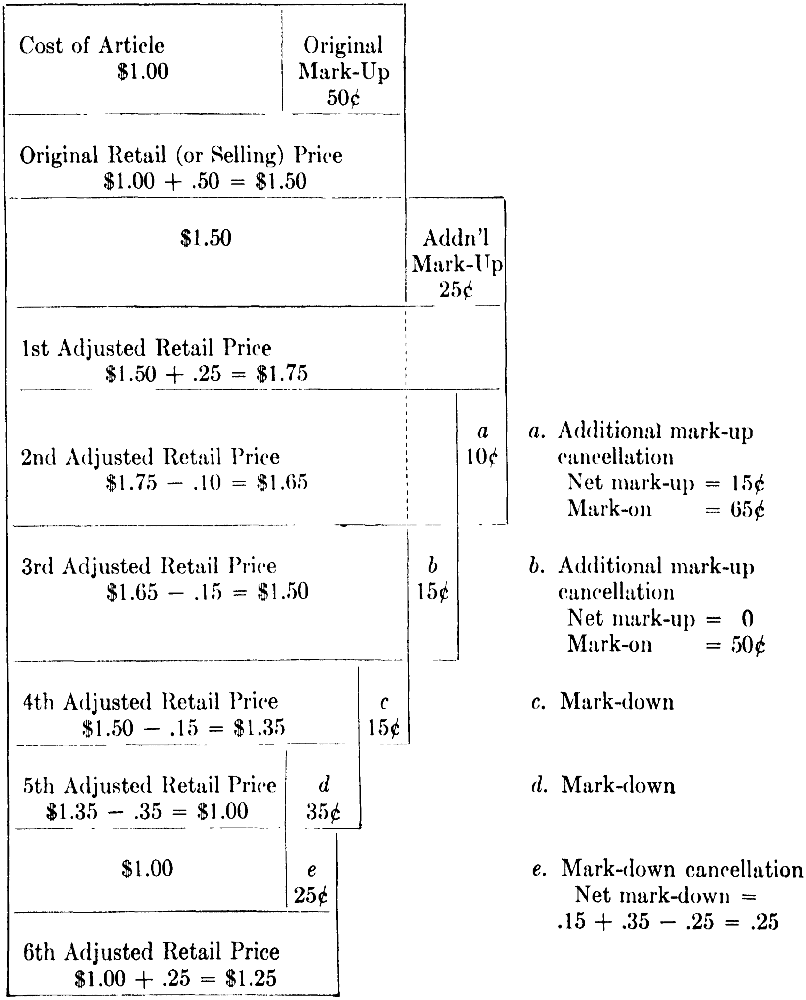
Mark-downs. Deductions from the original retail price to establish a new but lower retail price are mark-downs.

Mark-down cancellations. A reduction of the amount of a mark-down is a mark-down cancellation. Mark-down cancellations cannot exceed the total mark-downs. It is evident that the retail price of merchandise is increased when the mark-down is reduced, but such an increase is not to be considered as an additional mark-up.

Net mark-down. The difference between the sum of the mark-downs and the sum of the mark-down cancellations is the net mark-down.

Mark-on. The difference between cost and the original retail plus the net mark-up is the mark-on.

To illustrate the terms, let the following transactions be assumed.



Determining the ratio of cost to retail. In determining the ratio of cost to retail, it is customary to include additional mark-ups and additional mark-up cancellations but to exclude mark-downs and mark-down cancellations. To illustrate, let us assume the following facts:

Inventory at beginning of month:	
Cost	$30,000.00
Retail	43,000 00
Purchases:	
Cost	46,000 00
Retail	55,000.00
Returned purchases:	

Cost	1,000 00
Retail	1,500 00
Additional mark-ups	5,500 00
Additional mark-up cancellations	2,000 00
Mark-downs	6,000 00
Mark-down cancellations	1,000 00
Sales at retail	71,000.00

Compute the inventory by the retail method.

Solution

	Cost	*Retail*
Inventory	$30,000 00	$ 43,000 00
Purchases	46,000 00	55,000 00
	$76,000 00	$ 98,000 00
Deduct: Returned purchases	1,000 00	1,500 00
	$75,000.00	$ 96,500 00
Additional mark-ups less cancellations thereof		3,500 00
		$100,000 00

$100,000 − $75,000 = $25,000.
$25,000 ÷ $100,000 = 25%.

Mark-downs less mark-down cancellations	5,000 00
	$ 95,000 00
Sales at retail	71,000 00
End-of-month inventory at retail value	$ 24,000.00

$24,000 × 25% = $6,000.
$24,000 − $6,000 = $18,000, the cost value of the inventory.

Problems

1. From the records kept for Department B, the following information is obtained:

	Cost	*Retail*
Inventory at Beginning of Month	$15,000 00	$25,000 00
Purchases	36,000 00	54,000 00
Returned Purchases	500.00	700 00
Additional Mark-Ups		2,000 00
Additional Mark-Up Cancellations		1,000 00
Sales		60,000 00

Calculate by the retail method of inventory the cost of the book inventory at the end of the month.

2.* In a certain department of a large dry-goods house, the purchases for one year were $30,000. They were in the first place marked up for selling purposes to $45,000. Later, additional mark-ups amounting to $2,000 were made, and mark-downs aggregating $5,000 were also recorded. At the end of the fiscal period there were found to be on hand goods of a marked selling value of $10,000. State how you would arrive at their inventory value for the purpose of closing the books, and calculate the amount. Explain fully.

* American Institute Examination.

CHAPTER 18

Gross Profit Computations

Gross profit. The gross profit represents the margin between the sales and the cost of goods sold, and when expressed as a per cent of sales indicates to one who is familiar with trade practice whether a sufficient margin of profit is being made. Use of the per cent of gross profit to check the correctness of the value set upon the inventory is called the *gross profit test of inventory.*

Rate per cent of gross profit. The gross profit test is based on the supposition that in normal times and under normal conditions, any business will produce approximately the same per cent of gross profit on sales in any one period of time as in any other corresponding period of time.

Procedure. Statements of the gross profit and sales for each of several prior periods should be obtained. The gross profit for any one period divided by the sales for the same period gives the rate of gross profit for that period, *based on sales.* Disregard any per cent that is abnormal. Add the remaining per cents, and divide by the number added. The quotient is the average per cent of gross profit in prior periods.

Uses. The per cent of gross profit may be used in two ways: first, to prove inventories; and second, to compute the estimated inventory when it is impossible or impracticable to take a physical inventory.

Example

Assume that the average gross profit for the past five years has been 40% of sales, and that an audit of the books shows that the inventory, taken prior to the beginning of the audit, and valued at $100,000, seems smaller than it should be, while the previous inventory and purchases amounted to $400,000. The sales for the period are $400,000. Show by comparative statement the possibility of error.

Solution

In the following set-up, both the average and the current per cents and results are shown. As the profit in prior years has been 40% of sales, the cost of goods sold has been 60% of sales. 60% of $400,000 (sales) = $240,000, cost of sales.

		Current Year		Current Year in Terms of an Average Year	
		Actual Amounts	*Current Per Cent*	*Test Amounts*	*Average Per Cent*
Sales		$400,000	100	$400,000	100
Cost of Sales					
First of year inventory and purchases	$400,000				
Less: Current inv't.	100,000	300,000	75	240,000	60
Gross profit		$100,000	25%	$160,000	40%

If the sales are correct, the cost of sales is $60,000 too high, unless the rate has really changed. This discrepancy may be caused by any of the following: the volume of sales may be incorrectly stated; the current inventory may be erroneous, and the cost of sales affected thereby; or there may be an abnormal increase in the cost of merchandise purchased, when compared with the 5-year average. The accountant should determine the reason for the discrepancy.

Cost of goods sold. The average rate per cent of gross profit, applied to the sales for the current period, will give the estimated gross profit for the current period. Deduction of the estimated gross profit from the sales gives the estimated cost of goods sold. This procedure may be reduced to a formula as follows:

Average for Prior Periods

1. *Sales − Cost of sales = Gross profit.*
2. *Gross profit ÷ Sales = Per cent of gross profit (based on sales).*

Application to Current Period

3. *Sales × Per cent of gross profit (prior periods) = Estimated gross profit.*
4. *Sales − Estimated gross profit (current period) = Estimated cost of sales.*

Example

	Sales	*Cost of Sales*	*Gross Profit*
First period	$400,000	$300,000	$100,000
Second period	450,000	340,000	110,000
Third period	350,000	260,000	90,000
Fourth period	100,000	....	....

What was the cost of sales during the fourth period?

Solution

Average for Prior Periods

First period,	$100,000 ÷	$400,000	=	25.00%
Second period,	110,000 ÷	450,000	=	24 44%
Third period,	90,000 ÷	350,000	=	25.71%
				75.15%

75% ÷ 3 = 25%, the average rate of gross profit.

APPLICATION TO CURRENT PERIOD

$100,000 × 25% = $25,000, estimated gross profit.
$100,000 − $25,000 = $75,000, estimated cost of sales.

Rate per cent of cost of sales. If the rate of profit has been based on cost price instead of on selling price, the cost of sales may be tested by the following computations:

AVERAGE FOR PRIOR PERIODS

1. *Sales − Cost of sales = Gross profit.*
2. *Gross profit ÷ Cost of sales = Per cent of gross profit (based on cost of sales)*

APPLICATION TO CURRENT PERIOD

3. *Sales ÷ (100% + Per cent of gross profit) = Cost of sales.*

Example

	Sales	*Cost of Sales*
First period	$400,000	$300,000
Second period	450,000	340,000
Third period	350,000	260,000
Fourth period	100,000	....

What was the cost of sales for the last period?

Solution

AVERAGE FOR PRIOR PERIODS

First period,	$100,000 ÷ $300,000 =	33 33%
Second period,	110,000 ÷ 340,000 =	32 35%
Third period,	90,000 ÷ 260,000 =	34 61%
		100.29%

100% ÷ 3 = $33\frac{1}{3}$%, average per cent of gross profit.

APPLICATION TO CURRENT PERIOD

$100,000 ÷ 1.33$\frac{1}{3}$ (1 + .33$\frac{1}{3}$) = $75,000, cost of sales.
$100,000 − $75,000 = $25,000, gross profit.

It follows that if the cost of sales can be found, any element (inventory at beginning of period, purchases, closing inventory, and so forth) which goes to make up the cost of sales can be found, provided the other elements of the costs are given.

Fire losses. Insurance companies are generally willing to settle inventory losses resulting from fire on the basis of values determined by the gross profit method.

Example

The insurance company agrees that the following facts are to be the basis of its reimbursement to the insured for his fire losses:

Average gross profit for 4 years, 40% of sales.
Sales for this period to date of fire, $50,000.
Cost of goods available for sale, $300,000.

Solution

$50,000 (sales) × 40% (rate of gross profit) = $20,000, gross profit.
$50,000 (sales) − $20,000 (gross profit) = $30,000, cost of goods sold.
$300,000 (goods available for sale) − $30,000 (cost of goods sold) = $270,000, estimated inventory at date of fire.

Use of gross profit test in vertification of taxpayer's inventory. Assessors make use of the gross profit test to determine the approximate inventory and to check the item of inventory in the schedule filed by the taxpayer, since assessment dates seldom coincide with closing dates. The following forms have been given to the taxpayer to fill out, the date of assessment being May 1.

For Merchants

1. Book value of last inventory of stock of merchandise ....
2. Add purchases since last inventory to May 1 ... .
3. Add in-freight and cartage paid since last inventory to May 1 .. ..
4. Total of above three items ....

Deduct from above total net result of following two items:

5. Amount of net sales from date of last inventory to May 1 ....
6. Less. Gross profit on sales estimated at ......% ....
   (Previous near % may be used where actual % is unknown.)
7. Net inventory of merchandise on May 1 (Item 4 less Item 6) .. ..

For Manufacturers

1. Book value of raw materials, finished goods, and work-in-process at last inventory. Date ....
2. Add purchases of raw materials and finished goods since last inventory to May 1 ....
3. Add amount paid for in-freight and cartage from last inventory to May 1 .. ..
4. Add amount paid for labor and manufacturing expenses from last inventory to May 1 ....
5. Total of above four items ....

Deduct from above total the net result of the following two items:

6. Amount of net sales from date of last inventory to May 1 ....
7. Less: Gross profit on sales estimated at ......%. ....
   (Previous year % may be used where actual % is not known.) ....
8. Net value of raw materials, goods-in-process, and finished goods on May 1 (Item 5 − Item 7) .. ..

Problems

1. From the figures in the following tabulation, calculate the per cent of gross profit for each year, and by means of the average per cent of gross profit calculate the inventory at the end of the first half of the fifth year.

	Sales	Purchases	Opening Inventory	Closing Inventory	Per Cent of Gross Profit
First year	$120,000	$ 90,000		$10,000	....
Second year	150,000	100,000	10,000	12,000	....
Third year	165,000	110,000	12,000	10,000	....
Fourth year	180,000	122,000	10,000	11,000	....
Fifth year (6 mo.)...	95,000	62,000	11,000	....	....

2. From the following facts, find the inventory as of December 31:

Inventory, January 15 following, $16,578.50.
Sales, December 31 to January 15, $2,890.00.
$765 of the above sales shipped and invoiced before December 31.
Purchases, December 31 to January 15, at cost, $1,256.50.
Average gross profit, 25% of cost.

3. The average gross profit of the X. Company for the past three years has been 45% of the sales. During the fourth year the sales amounted to $159.500. Goods were purchased to the amount of $105,000. Returned purchases totaled $5,000 for the period. Freight paid on purchases was $6,000. The inventory at the beginning of the period was $40.000. Current market prices are 10% above the purchase prices for the year. Find the cost of replacing the goods at the end of the year.

4. On April 30, the board of managers of the Ames Mercantile Company removed the superintendent on the general suspicion that his books misrepresented the true financial condition of the business. Prepare a statement showing the nature and the probable extent of the misrepresentations; also an approximate statement of income and profit and loss for the four months ending April 30.

The following is a trial balance taken from the books, April 30:

Capital Stock		$ 75,000
Furniture and Fixtures	$ 10,000	
Inventory, January 1	128,600	
Cash	15,450	
Accounts Payable		39,000
Accounts Receivable	24,600	
Loans Payable		10,000
Sales		51,000
Purchases	40,700	
Salaries, Salesmen	2,200	
Advertising	1,650	
Salaries, Office	1,100	
Rent	400	
Interest	200	
Insurance, January 1 to December 31.........	999	
Stationery and Printing	105	
Reserve for Depreciation of Furniture & Fixtures............		2,710
Surplus, January 1......................		48,294
	$226,004	$226,004

An analysis of the Purchases, Sales, and Inventory accounts revealed the following:

	Purchases	*Sales*	*Opening Inv't*	*Closing Inv't*
First year	$122,000	$153,750	$101,000	$100,000
Second year	123,000	153,170	100,000	102,000
Third year	121,000	154,722	102,000	128,600

5.* The books of a concern recently burned out contained evidence of purchases, including inventory, to the amount of $200,000, and sales of $40,800, since the last closing. Upon investigation, however, the auditor ascertained that a sale of merchandise had been made just prior to the fire, and not recorded in the books, at an advance of two-fifths over cost less a 10% cash discount; the profit on the transaction was $31,928. The past history of the business indicated an average gross profit of 50% on cost of goods sold.

(*a*) What amount should be claimed as fire loss?

(*b*) What rate of gross profit do the transactions finally yield?

6.† The store and stock of the Diamond Jewelry Company was destroyed by fire on November 1. The safe was opened, and the books were recovered intact. The trial balance taken off was as follows:

Cash in Bank	$ 1,000	
Accounts Receivable	10,000	
Accounts Payable		$ 30,000
Merchandise Purchases..........	90,000	
Furniture and Fixtures..........	7,500	
Sales..........................		110,000
General Expense................	18,000	
Insurance......................	1,500	
Salaries.......................	5,500	
Real Estate—Store Lot..........	50,000	
Store Building.................	35,000	
Capital Stock..................		50,000
Surplus........................		28,500
	$218,500	$218,500

The average gross profit as shown by the books and accepted by the insurance companies was 40% of sales. The insurance adjuster agreed to pay 75% of the book value of furniture and fixtures, 90% of the book value of the store building, and the entire loss on merchandise stock.

Draft journal entries to include the account against the insurance companies.

Installment sales of personal property. The large increase in sales of personal property on the installment plan, and the option that the government allows a taxpayer coming within the definition of an installment dealer to return his gross income from sales on the installment basis, are indications of the growing importance of this subject.

The installment plan of selling was devised for the purpose of

* American Institute Examination.

† C. P. A., Oklahoma.

stimulating sales, whereas the installment basis of reporting income was devised for the purpose of deferring from year to year the income to be realized from installment sales, with a view to the possible effect that this deferment might have upon the amount of federal income tax to be paid. It is, of course, essential that the latest Federal Income Tax Law be observed.

Computation of gross profit. The gross profit to be reported may be ascertained by taking that proportion of the total cash collections received in the taxable year from installment sales (such collections being allocated to the year against whose sales they apply) which the annual gross profit to be realized on the total installment sales made during each year bears to the gross contract price of all such sales made during that particular year.

Example

The books of the Model Credit Company, selling merchandise on the installment plan, show the following:

	First Year	*Second Year*	*Third Year*	*Fourth Year*
Sales	$ 80,000	$110,000	$130,000	$90,000
Cost of Sales				
Inventory (old)	$ 45,000	$ 40,000	$ 50,000	$48,000
Purchases	55,000	75,000	90,000	50,000
	$100,000	$115,000	$140,000	$98,000
Less: Inventory (new)	40,000	50,000	48,000	40,000
Cost of sales	$ 60,000	$ 65,000	$ 92,000	$58,000
Gross profit	$ 20,000	$ 45,000	$ 38,000	$32,000

Collections were made in the fourth year on each year's contracts as follows:

First Year	*Second Year*	*Third Year*	*Fourth Year*
$1,600	$4,800	$25,000	$70,000

What was the gross profit to be reported for the fourth year?

Solution

Per Cent of Gross Profit

First year,	$20,000 (gross profit)	÷	$ 80,000 (sales)	=	25.00%
Second year,	45,000 " "	÷	110,000 "	=	40 91%
Third year,	38,000 " "	÷	130,000 "	=	29.23%
Fourth year,	32,000 " "	÷	90,000 "	=	35.55%

Profit on Collections in Fourth Year

Collected on first-year contracts,	$ 1,600 × 25.00% =	$ 400 00
" " second-year "	4,800 × 40.91% =	1,963 68
" " third-year "	25,000 × 29.23% =	7,307.50
" " fourth-year "	70,000 × 35.55% =	24,855.00
Gross profit realized in the fourth year		$34,526.18

Reserve ıor unearned gross profit. The gross income to be realized on installment sales is credited to "Reserve for Unearned Gross Profit," and at this time this account is debited with the gross profit on collections. The balance of the account represents gross profit on installment sales contracts remaining unpaid at the date of closing.

Example

The books of the X.Y.Z. Company, selling merchandise on the installment plan, show the following:

	First Year	*Second Year*	*Third Year*	*Fourth Year*
Sales	$89,257 99	$111,825 86	$137,012 32	$97,912 26
Gross profits	29,962 89	48,068 37	38,128 63	39,168 71
Collections during the fourth year on each year's accts	1,635.35	4,832 00	25,182.14	69,927 92

What amount should be credited to Reserve for Unearned Gross Profit to represent deferred income for the fourth year? What amount should be debited to Reserve for Unearned Gross Profit to represent income realized from the first, second, third, and fourth years' collections received in the fourth year?

Solution

(a) Per Cent of Gross Profit

First year	$29,962.89 ÷	$ 89,257.99 =	33 57%
Second year	48,068.37 ÷	111,825.86 =	42 98%
Third year	38,128.63 ÷	137,012.32 =	27 83%
Fourth year	39,168.71 ÷	97,912.26 =	40 00%

Profit on Collections

First-year accounts	$ 1,635.35 × 33.57%	= $ 548 99
Second-year accounts	4,832.00 × 42.98%	= 2,076 79
Third-year accounts	25,182.14 × 27.83%	= 7,008 19
Fourth-year accounts	69,927.92 × 40.00%	= 27,971.17
Gross profit realized in 4th yr		$37,605 14

Journal entries

Installment Sales Contracts	$ 97,912.26	
Cost of Sales		$ 58,743 55
Reserve for Unearned Gross Profit		39,168 71
Cash	$101,577.41	
Installment Sales Contracts		$101,577.41
Reserve for Unearned Gross Profit	$ 37,605.14	
Realized Gross Profit on Installment Sales		$ 37,605.14

Bad debts. The bad debts written off during the year should be allocated by years, and a charge should be made to Reserve for Unearned Gross Profit for the percentage of gross profit in each year's write-off, and to Profit and Loss (Bad Debts) for the remainder, the entire credit being made to Installment Sales Contracts.

Example

During the fourth year, bad accounts were written off as follows:

First-Yr. Accts.	*Second-Yr. Accts.*	*Third-Yr. Accts.*	*Fourth-Yr. Accts.*
$67 65	$141 05	$65 62	$126 25

What amount should be charged to these accounts: Profit and Loss (Bad Debts), and Reserve for Unearned Gross Profit?

Solution

	Unrealized Profit	*Remainder*
$ 67.65 × 33.57%	$ 22 71	$ 44 94
141.05 × 42.98%	60.62	80 43
65.62 × 27.83%	18 26	47.36
126.25 × 40.00%	50 50	75 75
	$152.09	$248 48

Reserve for Unearned Gross Profit	$152 09	
Profit and Loss (Bad Debts)	248 48	
Installment Sales Contracts		$400 57

Problems

1. The X.Y.Z. Company's books for the 5th year showed:

Sales	$128,642 60
Gross profit	42,975 12

Collections were made in the fifth year on each year's contracts as follows:

1st Yr.	*2nd Yr.*	*3rd Yr.*	*4th Yr.*	*5th Yr.*
$230 60	$1,590 31	$9,326 80	$21,256 30	$82,327 58

Calculate: (*a*) the per cent of gross profit for the fifth year; (*b*) the amount to be credited to Reserve for Unearned Gross Profit; (*c*) the amount to be debited to Reserve for Unearned Gross Profit. Use the rates given in the solution on page 162 for the first four years.

2. The analysis of bad debts written off during the 5th year was:

1st-Yr. Acct.	*2nd-Yr. Acct.*	*3rd-Yr. Acct.*	*4th-Yr. Acct.*	*5th-Yr. Acct.*
$8 35	$209 75	$910.40	$150 80	$470 62

What amounts should be charged to Reserve for Unearned Gross Profit and to Profit and Loss, respectively?

3. Results for the 6th year:

Sales	$140,695 39
Gross profit	54,541 07

Collections were made in the sixth year on each year's contracts as follows:

1st Yr.	*2nd Yr.*	*3rd Yr.*	*4th Yr.*	*5th Yr.*	*6th Yr.*
$62.70	$492.54	$2,798.30	$4,689.30	$2,657 80	$90,275.89

(*a*) Calculate the per cent of gross profit for the 6th year.
(*b*) Calculate for the 6th year the gross profit on collections made.

4. Accounts receivable were written off as follows:

1st Yr.	*2nd Yr.*	*3rd Yr.*	*4th Yr.*	*5th Yr.*	*6th Yr.*
$52.83	$31.5C	$51.10	$150.00	$163.82	$108.28

Compute the charges to be made to Profit and Loss (Bad Debts) and to Reserve for Unearned Gross Profit.

5.* The "A & B" Company is engaged in the business of retailing musical merchandise. The majority of the sales consist of installment sales of pianos and talking machines, on which the initial payment is less than 25% of the sales price and the balance is payable in monthly installments over a period of three to five years. The company was incorporated and began business on January 1. The following schedules are submitted on the various classes of merchandise:

	Piano Installment Sales	*Machine Installment Sales*	*Other Mdse Sales*
SALES			
First year	$148,650 00	$92,475 00	$38,337 60
Second year	163,520 00	88,535 00	39,543 50
Third year	180,400 00	94,256.00	40,731.15
PURCHASES			
First year	106,322 37	67,432.18	27,108 88
Second year	120,987 41	55,116 92	27,224 35
Third year	140,125 25	60,013.22	27,469 33
INVENTORIES			
First year	20,103 14	10,248 31	8,323 64
Second year	32,105 86	15,012 83	15,299 41
Third year	39,294 44	18,144 77	13,521 31

Attention is called to the fact that "Other Merchandise Sales" are sales for cash, or credit sales other than installment sales.

No adjustments to Deferred Income account are made until the end of the year. Additions to this income are made at the end of the year on the basis of the balance due on the current year's installment sales, and deductions are made on the basis of cash received during the current year on installment sales of previous years. On December 31 of the third year, the unpaid balances on third-year piano installment sales amount to $110,425.50, and on third-year machine installment sales to $60,475.00—exclusive of accrued interest. The following amounts were received during the third year on installment sales of previous years:

On first-year piano installment sales	$30,285.00
On second-year piano installment sales	42,413 00
On first-year machine installment sales	25,386.00
On second-year machine installment sales	26,285.00

The above amounts are exclusive of interest, which is credited direct to interest revenue.

Fractional percentages may be disregarded in the computation of ratios—over ½ of 1% should be added, and less than ½ of 1% should be dropped.

On first-year machine installment sales, uncollectible balances amounting to $399.00 were charged on the books of the company to expense, and credited to installment sales contracts.

Federal income taxes paid in the third year were charged to surplus.

* C. P. A., Michigan.

Depreciation is calculated at the following rates:

Buildings	2%	Furniture and Fixtures	10%
Auto Trucks	25%		

The following is a copy of the trial balance as of December 31, end of third year, before closing and before apportionment of deferred income on installment sales:

Cash	$ 15,327.48	
Notes Receivable	2,000.00	
Accounts Receivable	20,842.11	
Installment Sales Contracts	265,418.50	
Inventories	62,418.10	
War Bonds	5,000.00	
Real Estate	10,000 00	
Buildings	40,000 00	
Furniture and Fixtures	4,500 00	
Auto Trucks	3,000.00	
Notes Payable		$ 50,000.00
Accounts Payable		13,458 25
Deferred Income on Installment Sales		83,245 70
Reserve for Depreciation, Buildings		1,600 00
Reserve for Depreciation, Fur. & Fix		900 00
Reserve for Depreciation, Trucks		1,500 00
Capital Stock		150,000.00
Surplus		75,556.21
Sales		315,387.15
Piano Rentals		1,785 00
Interest on Installment Sales		2,035 23
Interest on War Bonds		237 50
Cash Discounts on Purchases		2,452.07
Purchases	227,607.80	
Salaries, Officers	14,000 00	
Salaries, Store	8,101 46	
Light and Heat	717.68	
Advertising	4,015 71	
Truck Expense	508.53	
Sundry Store Expense	2,239 17	
Salaries, Office	2,020 00	
Traveling Expense	648 50	
Postage	472 30	
Telephone and Telegraph	441 40	
Insurance	1,309 06	
Real Estate and Personal Property Taxes	2,029 69	
Bad Debts, Accounts Receivable	709 66	
Bad Debts (first year machine installment sales)	399 00	
Repairs, Sundry	365.68	
Donations	200 00	
Cash Discounts on Sales	444 48	
State Franchise Tax	187.80	
Capital Stock Tax	233.00	
Interest Paid	3,000 00	
	$698,157.11	$698,157.11

You are asked to give: (*a*) the net taxable income (for federal tax purposes) for the third year; (*b*) a balance sheet of the "A & B" Company as of January 1, beginning of fourth year.

Deferring income; its effect on tax. The statement was made in the second paragraph of this subject that the installment basis of accounting defers income with a view to the possible effect that deferment may have on the amount of federal income tax to be paid. Since the income is deferred, the tax is deferred (not saved).

The amount of profit realized and to be realized from the sales of a particular year, if not taxed in that particular year, will be taxed eventually, and the saving of tax results from a possible reduction in the rate of tax or from the spreading of taxable income over several years. If it is anticipated that the rate of tax will be increased, it may not be wise to defer the income.

Second, a change from the accrual to the installment basis results in double taxation, for Section 44 (c) of the Internal Revenue Code provides as follows: "If a taxpayer entitled to the benefits of subsection (a) elects for any taxable year to report his net income on the installment basis, then in computing his income for the year of change or any subsequent year, amounts actually received during any such year on account of sales or other dispositions of property made in any prior year shall not be excluded."

The amount of gross income which may be deferred on installment sales is governed by:

(1) The terms of sale;
(2) Annual increase, if any, in sales;
(3) Per cent of year's sales collected in the current year; and
(4) Fluctuation of gross profits.

Example

Assume the terms of sale to be 10% down, and 10% a month; the annual increase in sales to be $10,000; the per cent of year's sales collected, and the sales throughout the year, to be uniform; and the per cent of gross profit to be fixed.

Year	*Sales*	*Gross Profit on Sales*
First	$50,000	30%
Second	60,000	30%
Third	70,000	30%
Fourth	80,000	30%
Fifth	90,000	30%

Since it has been assumed that the sales are uniform throughout the year and that collections are met promptly, the second year's business may be analyzed as follows:

	Down Payments		*Installment Payments*	
Jan	10% of $ 5,000 =	$ 500		
Feb	10% of 5,000 =	500	10% of $ 5,000 =	$ 500
Mar	10% of 5,000 =	500	10% of 10,000 =	1,000
Apr	10% of 5,000 =	500	10% of 15,000 =	1,500
May	10% of 5,000 =	500	10% of 20,000 =	2,000
June	10% of 5,000 =	500	10% of 25,000 =	2,500
July	10% of 5,000 =	500	10% of 30,000 =	3,000
Aug	10% of 5,000 =	500	10% of 35,000 =	3,500
Sept	10% of 5,000 =	500	10% of 40,000 =	4,000
Oct	10% of 5,000 =	500	10% of 45,000 =	4,500
Nov	10% of 5,000 =	500	10% of 45,000 =	4,500
Dec	10% of 5,000 =	500	10% of 45,000 =	4,500
Year's sales	$60,000			
Down payments		$ 6,000		
Install. payments		31,500		$31,500
Total payments		$37,500		

Ratio of payments to sales: $37,500 ÷ $60,000 = 62.5%.

A comparison of the income to be reported on the accrual basis and on the installment basis may be made as follows:

ACCRUAL BASIS

	Second Year	*Third Year*	*Fourth Year*	*Fifth Year*
Sales	$60,000	$70,000	$80,000	$90,000
Gross profit (%)	30%	30%	30%	30%
Gross profit ($)	18,000	21,000	24,000	27,000

INSTALLMENT BASIS

Collections:				
1st-year accounts	$18,750			
2nd-year accounts	37,500	$22,500		
3rd-year accounts		43,750	$26,250	
4th-year accounts			50,000	$30,000
5th-year accounts				56,250
Gross income to be reported:				
30% of 1st-year coll	$ 5,625			
30% of 2nd-year coll	11,250	$ 6,750		
30% of 3rd-year coll		13,125	$ 7,875	
30% of 4th-year coll			15,000	$ 9,000
30% of 5th-year coll				16,875
Total income reported	$16,875	$19,875	$22,875	$25,875
Income deferred	$ 1,125	$ 1,125	$ 1,125	$ 1,125

It may be observed from the foregoing analysis that with an annual increase of $10,000 in sales, and with a constant gross profit ratio of 30%, the amount of income deferred from year to year is $1,125.

With an annual increase of $20,000 in sales, and other condi-

tions the same, the amount of income deferred would be $2,250 (2 × $1,125).

Problems

1. Assume the terms of sale to be 10% down and 5% a month, the annual increase in sales $10,000, the per cent of year's sales collected and the sales throughout the year uniform, and the per cent of gross profit fixed.

Year	*Sales*	*Gross Profit on Sales*
First	$50,000	30%
Second	60,000	30%
Third	70,000	30%
Fourth	80,000	30%
Fifth	90,000	30%

Show the amount of income deferred when the installment basis is used.

2. If the terms of payment were 5% down and 5% a month, and other conditions were the same as in Problem 1, what would be the amount of income deferred each year?

CHAPTER 19

Analysis of Statements

Financial and operating ratios. An analysis of the financial and the operating ratios of a business means a study of the relationships that are expressed in the statistics presented. Well-known and commonly used ratios are those of expenses and earnings to sales, and of earnings on capital employed. Other ratios, relationships, and turnovers that are indicators of the condition of a business should also be considered.

A summary of financial and operating ratios, relationships, and turnovers would include the following:

(1) Ratio of costs and expenses to net sales.
(2) Ratio of gross profit to net sales.
(3) Ratio of operating profit to net sales.
(4) Ratio of net profit to net sales.
(5) Ratio of operating profit to total capital employed.
(6) Ratio of net profit to net worth.
(7) Earnings on common stockholders' investments.
(8) Working capital ratio.
(9) Sources of capital.
(10) Manner in which capital is invested.
(11) Turnover of total capital employed.
(12) Turnover of inventories.
(13) Turnover of accounts receivable.
(14) Turnover of fixed property investment.

There are many other ratios which are important measures of efficiency, but of which only brief mention can be made in this chapter. Depending on the type of business being analyzed, these other ratios might include the labor turnover, the unit of output per operative, the average wage per man, the average wage per hour, and other statistics.

Costs, expenses, and profits. Costs, expenses, and profits should be expressed as per cents of money values and, where possible, should be expressed in terms of dollars per production unit, such as the ton, pound, yard, or gallon. The per cents, compared with those of previous years, show whether sales prices have been

adjusted proportionately to costs of production and distribution. The unit prices supplement the per cents and afford a direct comparison.

Ratio of gross profit to net sales. The ratio of gross profit to net sales is an indication of the spread between the cost of production and the selling price. The gross profit must be as large as possible, for out of it must come the expenses of selling, administration, finance, and other charges, before a net return is realized on capital.

Ratio of operating profit to net sales. The ratio of operating profit to net sales expresses the basic relationship between profits and sales. Operating profits represent the gain before the deduction of federal taxes, interest on borrowed money, and extraordinary losses, but do not include miscellaneous income not attributable to ordinary operations.

Ratio of net profit to net sales. The ratio of net profit to net sales indicates the margin of profit on the selling price. The rapidity of stock turnover, and the capital invested in accounts receivable, in inventory, and in plant, should be considered with this ratio.

Ratio of operating profit to total capital employed. The ratio of operating profit to total capital employed forms a ready basis for a comparison of the operating results of a business or of several plants under a single control. Capital employed includes plant, inventories, accounts receivable, cash balances, and so forth, regardless of the source of such capital, and is readily determined by referring to the asset side of the balance sheet.

Ratio of net profit to net worth. The ratio of net profit to net worth expresses the measure of earnings available to the stockholders or proprietors, and is the final indicator of the success or failure of any business.

Earnings on common stockholders' investments. The earnings on common stockholders' investments are based on the stockholders' share of the net profit, in relation to their interest in the net worth of the business. There are two ways in which these earnings may be stated: (*a*) as a per cent of the amount of such investments; and (*b*) in dollars earned per share outstanding.

Example

The following profit and loss statement, together with certain other facts, is presented to illustrate items 1–7 in the summary on page 169. The numbers in parentheses refer to the numbered ratios in the summary.

BLANK MERCANTILE COMPANY

PROFIT AND LOSS STATEMENT
FOR THE TWELVE MONTHS' PERIOD ENDED DECEMBER 31, 19—.

Sales:					
Gross Sales		$693,004.10			
Less: Sales Rebates and Allowances ...	$ 870.64				
Prepaid Freight..	200.25				
		1,070 89			
Net Sales			$691,933 21	100.00%	
Cost of Sales:					
Inventory, beginning of year		$107,278 46			
Purchases......... ..	$624,225 28				
Freight........	16,271 98				
	$640,497 26				
Less: Purchase Rebates and Allowances	630 81				
		639,866 45			
		$747,144 91			
Inventory, end of year.		124,814 04			
Cost of Sales			622,330 87	89 94	(1)
Gross Profit			$ 69,602 34	10 06%	(2)
Delivery Expenses:					
Salaries of Drivers.....	$ 3,414 34				
Dep'n on Equipment ..	2,839 57				
Auto Repairs	1,562 53				
Gasoline and Oil	1,479 27				
Drivers' Expenses.....	119 40				
Drayage..............	66 84				
Total...............		$ 9,481.95		1.37	(1)
Selling Expenses:					
Salesmen's Salaries.....	$ 11,812 50				
Salesmen's Expenses...	1,942.06				
Advertising...........	844.32				
Telephone and Telegraph....	642.57				
Total...............		$ 15,241.45		2.20	(1)

General Expenses:					
Salaries	$ 8,722 33				
Expenses	613 36				
Executive Salaries	3,600 00				
Taxes (other than federal)	1,906 23				
Insurance	1,723 46				
Depreciation	1,259 54				
Light, Heat, and Water	829 49				
Printing and Stationery	444.50				
Postage	408.52				
Collections	219 76				
Repairs	115 91				
Storage	22 69				
Miscellaneous	380 50				
Total		20,246 29		2 93	(1)
Total Expense			44,969 69	6.50%	(1)
Net Operating Profit			$ 24,632 65	3 56%	(3)
Additions to Income:					
Discount on Purchases		$ 9,565 86			
Interest Earned		563 32			
Bad Debts Recovered		102 53			
Total			10,231.71	1.48	
			$ 34,864.36	5.04%	
Deductions from Income:					
Discount on Sales		$ 4,771.92			
Interest Paid for Money Borrowed		4,373 16			
Interest Paid on Building Contract		3,010.00			
Bad Debts Reserve		1,283.91			
Donations		162 00			
Total			13,600.99	1 97	
Net Profit			$ 21,263.37	3 07%	(4)

Supplemental

Total Capital Used (see Balance Sheet, below)	$276,317.34		
Ratio of Profit to Capital		7.69%	(5)
Net Worth (beginning of year)	124,252.36		
Ratio of Profit to Net Worth		17.11	(6)
Common Stock Outstanding	113,400.00		
Number of Shares ($50.00 par value)	2,268		
Per cent earned		18.75	(7)
Dollars earned per share		9.38	

Working capital ratio. This ratio is probably the best-known measure applied to financial statements, because more than any other it has been stressed by bankers and businessmen. It is computed by dividing the amount of the current assets by the amount of the current liabilities. If the quotient is 2, the current assets are said to be in a "2 to 1" ratio; that is, in a ratio of $2 of current assets to each $1 of current liabilities.

What the working capital ratio should be depends upon differences in types of business, location, and other factors, the effect of which is to vary somewhat the proportions involved. While some lines of trade may be expected to maintain a 2-to-1 ratio, others may necessitate a proportion as high as 10 to 1.

The rapidity with which receivables and inventory are turned is a factor bearing on the adequacy of the working capital ratio. With respect to accounts receivable, there is a range of turnover from 3 days in some of the retail chain stores to 80 or 90 days in coal and heavy manufacturing industries. The turnover of inventories is most rapid in such industries as slaughtering and meat packing, retail chain stores, chemical products, and iron and steel, while the turnover of inventories is found to be slow in such industries as tobacco products, machinery manufacturing, leather products, and rubber goods.

Example

The following balance sheet is presented to illustrate the working capital ratio. It will also be referred to in later paragraphs, where the computation of other ratios is discussed.

BLANK MERCANTILE COMPANY

BALANCE SHEET
DECEMBER 31, 19—.

Assets

Current:			
Cash in Banks	$ 13,598.85		
Cash on Hand	4,113.24	$ 17,712 09	
Accounts Receivable—Customers	$ 64,832 57		
Accounts Receivable—Others	647 92		
Notes Receivable—Customers	5,329 91		
Notes Receivable—Others	227.31		
Securities	1,274 34		
Accrued Interest	32 98		
Railroad Claims	93 76		
	$ 72,438.79		
Less: Reserve for Bad Debts	1,890.06	70,548 73	
Merchandise Inventory		124,814 04	
Total			$213,074.86
Fixed:			
Land		$ 3,450.00	
Warehouse Building	$ 50,373.48		
Warehouse Equipment	545.77		
Delivery Equipment	14,090.39		
Furniture and Fixtures	2,488.85		
	$ 67,498 49		
Less: Accumulated Depreciation	9,152.48	58,346.01	
Total			$ 61,796.01

Deferred Charges:			
Prepaid Insurance.		$ 1,298.13	
Prepaid Interest		148 34	
Total			1,446 47
			$276,317 34

Liabilities

Current:			
Payroll	$ 1,131.77		
Accounts Payable	16,177.08		
Notes Payable—Banks	50,000 00		
Notes Payable—Others	17,600 00		
Notes Payable—Stockholders	11,700 00		
Accrued Taxes	1,575 17		
Accrued Interest—Notes	1,393 92		
Accrued Interest—Contracts	3,010.00		
Total		$102,587.94	
Fixed:			
Warehouse Contract for Deed		43,000 00	
Net Worth:			
Capital Stock—Common	$113,400 00		
Surplus	17,329 40	130,729 40	
			$276,317 34

In the foregoing balance sheet, the current assets are stated at $213,074.86, and the current liabilities are stated at $102,587.94.

$$213{,}074.86 \div 102{,}587.94 = 2.077.$$

The ratio of working capital is, therefore, 2.077.

Sources of capital. The sources of capital may be stated in a general way under four headings, as follows:

(1) Short-term borrowings and credits.
(2) Long-term borrowings and credits.
(3) Stockholders' investments.
(4) Surplus (earnings left in the business).

Summarizing the liability section of the foregoing balance sheet and dividing each section total by the total of all sections, the ratio of capital supplied by each source is as shown in the right-hand column of the following tabulation:

	Amount	*Per Cent*
Current Liabilities	$102,587.94	37.13
Fixed Liabilities	43,000.00	15.56
Capital Stock—Common	113,400.00	41.04
Surplus	17,329.40	6.27
	$276,317.34	100.00

Manner in which capital is invested. The manner in which the capital is employed in the business is shown by a summary of the asset sections.

	Amount	*Per Cent*
Current Assets	$213,074.86	77.12
Fixed Assets	61,796.01	22.36
Deferred Charges	1,446.47	.52
	$276,317.34	100.00

Turnover of total capital employed. This item expresses the relation of the net sales to the total capital employed. The average capital employed throughout the year should be used, but, in the absence of monthly statements, the capital at the beginning of the year and the capital at the end of the year should be added and divided by two to give an estimate of the average capital employed. In arriving at this average, investments not employed in operations should be eliminated from the total assets, for, as a rule, they represent a surplus not required in the conduct of the business. Income from such investments should be eliminated from the statement of earnings before the ratio is computed.

Total assets at beginning of year	$246,351.89
Total assets at end of year	276,317.34
	2)$522,669 23
Average capital employed (securities not eliminated, as the amount was negligible)	$261,334.61

The turnover of total capital employed is, therefore:

$691,933.21 (net sales) ÷ $261,334.61 (average capital) = 2.64.

Turnover of inventories. The subject of inventory turnover was presented in Chapter 17.

The rate of turnover is computed as follows:

$622,330.87 (cost of sales) ÷ $112,131.69 (average inventory) = 5.55.

Turnover of accounts receivable. The normal credit period, whether it be 30, 60, or 90 days, is compared with the average number of days' sales uncollected obtained from the following formula, as a means of judging the efficiency of the collection department:

$$\frac{\textit{Accounts receivable at end of fiscal period}}{\textit{Sales for fiscal period}} \times \textit{Days in fiscal period}$$

$$= \textit{Average number of days' sales uncollected.}$$

The Accounts Receivable account showed $64,832.57 of outstanding accounts at the close of the fiscal period. The sales for the fiscal period of 12 months amounted to $691,933.21, and the

average term of credit granted at time of sale was 30 days. The average number of days' sales represented in standing accounts is computed as follows:

$$\frac{64{,}832.57}{691{,}933.21} \times 365 = 34.$$

If the average number of days' sales uncollected is greater than the average term of credit, the presence of overdue accounts is indicated. This is true of the example just given.

Turnover of fixed property investment. This turnover expresses the relationship between the volume of business done and the capital invested in plant and equipment. Large investments in property and equipment increase the expense burden through charges for depreciation, insurance, taxes, and so forth, and may make a favorable or an unfavorable operating statement, depending on the volume of business handled.

The number of dollars of sales for each dollar of fixed property investment is calculated as follows:

$691,933.21 (net sales) ÷ $58,346.01 (net fixed property investment) = 11.86.

Problems

1. From the balance sheets and supplemental information, determine the ratios named, following the balance sheets.

Assets	*This Year*	*Last Year*
Current Assets	$215,003 48	$213,074.86
Fixed Assets—Net	57,535 04	61,796.01
Deferred Charges	1,193.59	1,446.47
Total	$273,732.11	$276,317.34
Liabilities		
Current Liabilities	$ 86,229.30	$102,587.94
Fixed Liabilities	38,000 00	43,000 00
Total Liabilities	$124,229.30	$145,587.94
Net Worth		
Capital Stock	$124,300.00	$114,300.00
Surplus	25,202.81	16,429.40
Total Net Worth	$149,502.81	$130,729.40
Total	$273,732.11	$276,317.34
Annual Sales	$688,167.98	$691,933 21
Annual Expense	47,340.74	44,969.69
Ratios		
Current Ratio	....	....
Worth to Debt	....	....
Worth to Fixed Assets	....	....
Sales to Fixed Assets	....	....
Sales to Current Debt	....	....
Sales to Worth	....	....
Expense to Sales (%)	....	....

2. The United Manufacturing Company's card in the credit file of the Second National Bank contained the data for the year ended January 31, 1944, and from their balance sheet and profit and loss statement you have entered the comparative figures for the year ended January 31, 1945. Compute the comparative ratios for 1945.

COMPARATIVE RATIOS

	1/31 1944	1/31 1945	19	19	19
FIXED ASSETS TO TANGIBLE NET WORTH	24.8				
CURRENT DEBT TO TANGIBLE NET WORTH	48.0				
NET WORKING CAPITAL REP BY FUNDED DEBTS					
NET SALES TO INVENTORY	4.3				
NET WORKING CAPITAL REP BY INVENTORY	107.7				
INVENTORY COVERED BY CURRENT DEBT	61.3				
AVERAGE COLLECTION PERIOD	42.5				
TURNOVER OF TANGIBLE NET WORTH	3.4				
TURNOVER OF NET WORKING CAPITAL	4.7				
NET PROFITS ON NET SALES	.52				
NET PROFITS ON TANGIBLE NET WORTH	1.8				
NET PROFITS ON NET WORKING CAPITAL	2.4				
CURRENT ASSETS TO CURRENT DEBT	2 5				
TOTAL DEBT INCLUDING N W TANGIBLE NET WORTH	148.0				
	$	$	$	$	$
SALES	388,553	394,774			
EXPENSES					
NET PROFIT	2,048	1,664			
WORKING CAPITAL	83,022	71,523			
TANGIBLE NET WORTH	113,983	114,649			
FIXED ASSETS	28,244	49,244			
FUNDED DEBT		8,280			

COMPARATIVE FINANCIAL STATEMENTS

ASSETS	1/31 1944	1/31 1945	19	19	19
CASH	3,206	1,862			
ACCOUNTS RECEIVABLE	45,199	42,267			
NOTES TRADE ACCEPT RECV.					
INVENTORIES	89,342	83,218			
TOTAL CURRENT	168,709	127,349			
DUE FROM AFFILIATE OR SUBS'Y					
LAND BUILDINGS					
MACHINERY-FIXTURES	28,244	49,248			
NOTES ACC'TS (OFFICERS, PARTNERS)	2,716	2,156			
Organization Expense		999			
TOTAL ASSETS	168,709	179,754			
LIABILITIES					
ACCOUNTS PAYABLE-TRADE	34,037	32,671			
ACCEPT, NOTES PAYABLE					
BANKS PAYABLE	10,000	15,040			
PAYABLE AFFILIATE OR SUBS'Y					
ACCRUALS	2,557	2,793			
Due Officers	8,130	5,321			
TOTAL CURRENT	54,725	55,825			
MORTGAGES					
CHATTEL MORTGAGES					
Deferred Bank Loan		8,280			
TOTAL LIABILITIES	54,725	64,105			
CAPITAL STOCK	103,100	103,100			
SURPLUS	10,883	12,548			
TOTAL LIABILITIES & NET WORTH	168,709	179,754			

3. From the data given in the following balance sheet and profit and loss statement, together with the supplemental data, compute the fourteen financial and operating ratios relationships, and turnovers outlined in the preceding sections of this chapter.

BLANK MERCANTILE COMPANY

Balance Sheet
December 31, 19—

Assets

Current:			
Cash in Banks	$ 13,771 58		
Cash on Hand	3,616 34	$ 17,387.92	
Accounts Receivable—Customers	$ 59,424 48		
Accounts Receivable—Others	704 30		
Notes Receivable—Customers	3,746 76		
Notes Receivable—Others	272.19		
Securities	994 64		
Accrued Interest	52 30		
Railroad Claims	50 95		
	$ 65,245 62		
Less: Reserve for Bad Debts	3,852.57	61,393 05	
Merchandise Inventory		136,222 51	
Total			$215,003 48
Fixed:			
Land		$ 3,450.00	
Warehouse Building	$ 50,180 55		
Warehouse Equipment	545 77		
Delivery Equipment	14,090 39		
Furniture and Fixtures	2,503 85		
	$ 67,320 56		
Less: Accumulated Depreciation	13,235 52	54,085 04	57,535 04
Deferred Charges:			
Prepaid Insurance			1,193 59
Total			$273,732 11

Liabilities

Current:			
Payroll	$ 1,116.17		
Accounts Payable	13,325.73		
Notes Payable—Banks	24,000 00		
Notes Payable—Others	14,500 00		
Notes Payable—Stockholders	26,700 00		
Accrued Taxes	1,641 97		
Accrued Interest—Notes	2,285 33		
Accrued Interest—Contracts	2,660.00		
Total		$ 86,229.20	
Fixed:			
Warehouse Contract for Deed		38,000.00	
Net Worth:			
Capital Stock—Common	$123,400.00		
Surplus	26,102.91	149,502.91	
Total			$273,732.11

BLANK MERCANTILE COMPANY

Profit and Loss Statement
For the Year Ended December 31, 19—.

Sales:				
Gross Sales...............		$689,361.43		
Less: Sales Rebates and Allowances..........	$ 1,059 89			
Prepaid Freight.	133 56			
		1,193.45		
Net Sales			$688,167.98	100.00%
Cost of Sales:				
Inventory, beginning of year		$124,814.04		
Purchases....	$611,332.45			
Freight..................	15,184 68			
	$626,517.13			
Less: Pur. Rebates and Allowances..........	1,392 74			
		625,124.39		
		$749,938.43		
Inventory, end of year.....		136,222 51		
Cost of Sales.............			613,715 92	..........%
Gross Profit................			$ 74,452.06	..........%
Delivery Expenses:				
Salaries of Drivers.........	$ 3,874.27			
Dep'n on Equipment.......	2,818.08			
Auto Repairs	1,430.61			
Gasoline and Oil..........	1,231 29			
Drivers' Expenses.........	125 35			
Drayage..................	52 91			
Total.................		$ 9,532.51		..........%
Selling Expenses:				
Salesmen's Salaries........	$ 12,300 00			
Salesmen's Expenses.......	2,015 78			
Advertising...............	1,357.83			
Telephone and Telegraph...	536.21			
Total.................		16,209.82		..........%

General Expenses:				
Salaries	$ 8,797 50			
Expenses	265 43			
Executive Salaries	4,175 00			
Taxes (other than federal)	2,069 17			
Insurance	1,937 82			
Depreciation	1,264 96			
Light, Heat, and Water	826 33			
Printing and Stationery	516 70			
Postage	486 85			
Collections	238 65			
Repairs	106 47			
Storage	18 29			
Miscellaneous	895 24			
Total		$ 21,598.41		..........%
Total Expense			47,340 74	..........%
Net Operating Profit			$ 27,111 32	..........%
Additions to Income:				
Discount on Purchases		$ 9,759 20		
Interest Earned		1,348 60		
Bad Debts Recovered		10 65		
Total			11,118.45	..........%
			$ 38,229.77	..........%
Deductions from Income:				
Discount on Sales		$ 4,523.98		
Interest Paid for Money Borrowed		4,443.87		
Interest Paid on Building Contract		2,660 00		
Bad Debts Reserve		3,446 80		
Donations		269 20		
Total			15,343 85	..........%
Net Profit			$ 22,885.92	..........%

Supplemental

Total Capital Employed (see Balance Sheet)	$..............	
Ratio of Profit to Capital		..........%
Net Worth (beginning of year)	130,729.40	
Ratio of Profit to Net Worth		..........%
Common Stock Outstanding (see Balance Sheet)	....	
Number of Shares ($50.00 per value)	....	
Per Cent Earned		..........%
Dollars Earned Per Share		....

CHAPTER 20

Partnership

Definition. A partnership association is defined by Chancellor Kent as follows: "A contract of two or more competent persons to place their money, effects, labor, and skill, or some or all of them, in lawful commerce and business, and to divide the profits and bear the losses in certain proportions."

Mathematical calculations. The most important mathematical calculations in partnership accounting are concerned with:

(1) Division of profits.
(2) Division of assets upon liquidation.
(3) Calculation of goodwill.

Goodwill. The calculation of goodwill also has to be considered in connection with the other types of business organizations—namely, individual proprietorship and corporation—when changes in ownership, reorganizations, consolidations, and so forth, are made; see Chapter 21.

Profit-sharing agreements. Profits may be shared in many ways. A few of the most common methods of profit distribution are:

(1) Arbitrary ratios.

(2) In the ratio of capital invested at organization of business.

(3) In the ratio of capital accounts at the beginning or at the end of each period.

(4) In the ratio of average investments.

(5) Part of the profits may be distributed as salaries or as interest on capital invested, and the remainder in some other ratio.

(6) If the investment is less than the amount agreed upon, interest is charged on the shortage; and if the investment is more than the amount agreed upon, interest is credited on the excess; the resulting profit or loss is then distributed in a ratio agreed upon.

Lack of agreement. If the partners have failed to include in their articles of co-partnership an agreement as to the method by which profits are to be distributed, the law provides that the

profits shall be divided equally, regardless of the ratio of the partners' respective investments.

Losses. If losses are incurred and no provision has been made for their distribution, the profit-sharing ratio governs.

Arbitrary ratio.

Example

A and *B* are partners. *A* has $3,000.00 invested, while *B* has $2,500.00 invested. *A* is to receive $\frac{2}{3}$ of the profits, and *B* is to receive $\frac{1}{3}$. The profits for the year are $2,400.00. What is each partner's share?

Solution

Net profits	$2,400 00
A's share, $\frac{2}{3}$ of $2,400.00	1,600 00
B's share, $\frac{1}{3}$ of $2,400.00	800.00

Problems

A and *B* were partners. Gain or loss was to be divided $\frac{3}{5}$ and $\frac{2}{5}$, respectively. *A* invested $3,500.00, and *B* invested $2,400.00. During the year, *A* withdrew $500.00, and *B* withdrew $700.00. At the end of the year the books showed the following assets and liabilities:

Cash on Hand and in Bank	$8,000 00
Inventory of Merchandise	7,500 00
Notes Receivable	790 00
Accounts Receivable	840.00
Notes Payable	4,700 00
Accounts Payable	7,240 00

(*a*) What has been the gain or loss? (*b*) What is each partner's net capital at the end of the year?

Ratio of investment.

Example

January 1, *A*'s investment	$10,000.00
January 1, *B*'s investment	6,000 00
January 1, *C*'s investment	4,000 00
Total	$20,000.00
December 31, Profits	$ 4,000 00

Profits are to be shared in the ratio of investments at the beginning of the year.

Solution

	Investment	*Ratio*	*Profits*	*Shares*
A	$10,000	$\frac{10}{20}$	$4,000	$2,000
B	6,000	$\frac{6}{20}$	4,000	1,200
C	4,000	$\frac{4}{20}$	4,000	800
	$20,000	$\frac{20}{20}$		$4,000

Explanation. Add the beginning-of-year investments of each of the partners, and take for the numerator of the fraction representing each partner's share his investment at the beginning of the year, and for the denominator the total

capital. Using these fractions, calculate the fractional parts of the net profit or loss, and these will be the partners' shares.

Problems

In each of the following, show the division of net profit or net loss, which is to be calculated in the ratio of investments:

	Investments				
	A	*B*	*C*	Net Profit	Net Loss
1.	$4,000	$4,000	$2,000	$2,500	
2.	5,000	3,000	1,500		$1,200
3.	6,000	7,500	2,500	2,000	
4.	2,000	3,500	1,500	1,400	
5.	3,500	2,500	1,000	750	

Division of profits by first deducting interest on capital.

Example

January 1, *A*'s investment	$10,000
January 1, *B*'s investment	6,000
January 1, *C*'s investment	4,000
December 31, Net profits	4,000

By agreement, each partner is to receive 5% interest on his investment (this interest to be deducted from total profits), and the balance of the profits is to be distributed equally.

Solution

A's investment, $10,000, × .05	$ 500, interest
B's investment, 6,000, × .05	300, interest
C's investment, 4,000, × .05	200, interest
Total	$1,000

Net profits, $4,000 − $1,000 = $3,000, to be divided equally. $3,000 ÷ 3 = $1,000, each partner's share after interest is deducted.

	Interest	*Profit*	*Total Credit*
A	$500	$1,000	$1,500
B	300	1,000	1,300
C	200	1,000	1,200
Total			$4,000

Problems

Show the division of profits in each of the following:

	Investments			Net	Rate of Int. on	Balance to
	A	*B*	*C*	Profits	Investment	Be Divided
1.	$ 8,000	$ 4,250	$ 3,700	$4,000	5%	Equally
2.	9,750	3,500	10,000	6,000	6%	Equally
3.	4,725	5,300	5,250	5,300	6%	Equally
4.	12,000	6,000	4,000	4,500	4%	$\frac{4}{7}$, $\frac{2}{7}$, $\frac{1}{7}$
5.	20,000	10,000	5,000	5,000	6%	$\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$

Profits insufficient to cover interest on investment. If it is agreed that each partner is to be credited with interest on his

investment, the interest must be credited to each partner, even though the total profits are not large enough to cover the credit. Any over-distribution incurred by the distribution of the interest should be divided among the partners in accordance with the agreement as to the division of profits. The same rule applies where there is a loss before interest is credited.

Example

January 1, *A*'s investment	$10,000
January 1, *B*'s investment	6,000
January 1, *C*'s investment	4,000
December 31, Business profits	700

By agreement, each partner is to receive 5% interest on his investment, and the profits are to be shared equally.

Solution

A's investment, $10,000, × .05	$ 500, interest
B's investment, 6,000, × .05	300, interest
C's investment, 4,000, × .05	200, interest
Total interest to be credited	$1,000
Profits earned	700
Net loss	$ 300

Since the loss is to be shared equally, each partner's loss is $100.

	Credit Interest	*Debit Loss*	*Net Credit*
A	$500	$100	$400
B	300	100	200
C	200	100	100
Total			$700

Problems

Find the net credit or debit to each partner in each of the following:

	Investments *A*	*B*	*C*	Profit or Loss before Interest	Interest Rate	Balance to be Divided
1.	$ 8,000	$ 8,000	$ 4,000	Profit, $ 800	6%	Equally
2.	5,000	7,000	2,000	Profit, 140	6%	$\frac{5}{14}$, $\frac{7}{14}$, $\frac{2}{14}$
3.	3,800	4,200	5,000	Loss, 200	6%	Equally
4.	10,000	7,500	5,000	Profit, 2,150	6%	$\frac{5}{9}$, $\frac{3}{9}$, $\frac{1}{9}$
5.	15,000	15,000	10,000	Profit, 2,000	6%	Equally

Adjustments of capital contribution. If the partners do not invest the agreed amounts, adjustments may be made, provided the contract so states. Partners may be charged with interest on the amount of the shortage of their investment from the agreed amount, and may be credited with interest on the excess of their investment over the agreed amount. These adjustments should be made before the profits for the period are prorated. If interest adjustments result in an over-distribution of profits, the amount

over-distributed is divided in the ratio of the division of profits, unless otherwise agreed.

Example

	Agreed to Invest	Invested
January 1, *A*	$10,000	$12,000
January 1, *B*	6,000	5,000
January 1, *C*	4,000	2,000
December 31, Profits for the year		3,100

By agreement, *A* is to be allowed 5% interest on his excess investment, and *B* and *C* are to be charged 5% interest on their shortages. After these adjustments have been made, profits are to be divided equally.

Solution

A's excess, $2,000, × .05	$100, interest	
B's shortage, $1,000, × .05	50, interest	
C's shortage, $2,000, × .05	100, interest	
Charge to *B*'s account	$ 50	
Charge to *C*'s account	100	$150
Credit to *A*'s account		100
Net amount of interest		$ 50

The net amount of interest, $50, is added to net profits.
Profits before distribution:

Net profits	$3,100
Add net interest	50
Total	$3,150

$3,150 ÷ 3 = $1,050, each partner's share after interest adjustment.

A's $\frac{1}{3}$ profits	$1,050	
Add interest	100	$1,150, total credit of *A*
B's $\frac{1}{3}$ profits	$1,050	
Less interest	50	1,000, net credit of *B*
C's $\frac{1}{3}$ profits	$1,050	
Less interest	100	950, net credit of *C*
Total profits		$3,100

Problems

1. Prepare a statement of profit distribution from the following facts:

	A	*B*	*C*	*D*
Agreed investment	$6,000	$6,000	$8,000	$4,000
Investment	7,000	6,000	6,000	2,500
Profit ratio after adjustment of 6% interest on excess or deficiency of investment	25%	25%	$33\frac{1}{3}$%	$16\frac{2}{3}$%

Net profits before adjustments for interest, $6,500.

2. Prepare a statement of profit distribution from the following facts:

	X	*Y*	*Z*
Agreed investment	$5,000	$4,500	$4,500
Investment	4,000	4,000	6,000

Profits to be shared equally after adjustments for 6% interest. Profits before adjustments for interest, $600.

3. The capital of a certain organization was to be $40,000.00, of which A and B were to contribute one-half each, A to receive 55% of the profits and B to receive 45%. A, being short of funds, invested only $15,000.00, and, the firm being short of capital, B put in the balance until A could make up his shortage, with the provision that he be allowed 6% interest on the excess of his investment over the agreed amount. The profits for the year were $12,000.00. Show distribution of profits.

Profit sharing in ratio of average investment.

First method. Multiply the original investment by the number of days or months during which the amount was in the business without change. The product may be termed Day-Dollars or, Month-Dollars. The ratio of any product to the total of the products is the average capital ratio for that partner.

When the capital is changed, either by additional investment or by withdrawal, the changed capital is multiplied by the number of days or months to find its value in day-dollars or month-dollars, and for each change a new calculation is made. The ratio of the total of the day-dollars or month-dollars for each partner to the sum of the day-dollars or month-dollars for all the partners gives the ratio of each partner's investment to the total investment.

Example

A

Debit		*Credit*	
Feb. 1	$1,000	Jan. 1	$10,000
June 1	1,500	May 1	4,000
Nov. 1	500	July 1	1,000

B

July 1	$1,000	Jan. 1	$ 6,000
Dec. 1	1,000	Aug. 1	4,000
		Oct. 1	2,000

Net profits of the business for the year were $4,530.

Solution

A

	Invested	Months in Business	Month-Dollars	
Jan. 1,	$10,000	× 1 month	$10,000	
Feb. 1,	9,000	× 3 months	27,000	
May 1,	13,000	× 1 month	13,000	
June 1,	11,500	× 1 month	11,500	
July 1,	12,500	× 4 months	50,000	
Nov. 1,	12,000	× 2 months	24,000	
***A*'s month-dollars investment**		....		**$135,500**

B

Jan. 1, $ 6,000 × 6 months		$36,000	
July 1, 5,000 × 1 month		5,000	
Aug. 1, 9,000 × 2 months		18,000	
Oct. 1, 11,000 × 2 months		22,000	
Dec. 1, 10,000 × 1 month		10,000	
B's month-dollars investment			$ 91,000
Total month-dollars investment			$226,500
A's share of profits, $\frac{135,500}{226,500}$ of $4,530		$ 2,710	
B's share of profits, $\frac{91,000}{226,500}$ of $4,530		1,820	
		$ 4,530	

If the average investment is desired, it can be found by dividing the month-dollars by 12, as:

A's month-dollars, $135,500 ÷ 12	$11,291 67
B's month-dollars, $91,000 ÷ 12	7,583 33
Total average monthly investment	$18,875 00

The ratios of the average monthly investments are the same as the ratios of the month-dollars investments.

Second method. Multiply each investment by the number of months from the date made until the end of the period; find the sum of the products obtained. Likewise, multiply each withdrawal by the number of months from the date withdrawn until the end of the period; find the sum of the products obtained. Deduct the sum of the withdrawal products from the sum of the investment products; the result for each partner should be the same as the month-dollars obtained by the first method.

The example under the first method is used in the following solution.

Solution

A

Investments Date	Amount		Time to End of Year		Month-Dollars	
Jan. 1	$10,000	×	12 months	=	$120,000	
May 1	4,000	×	8 months	=	32,000	
July 1	1,000	×	6 months	=	6,000	
						$158,000
Withdrawals						
Feb. 1	$ 1,000	×	11 months	=	$ 11,000	
June 1	1,500	×	7 months	=	10,500	
Nov. 1	500	×	2 months	=	1,000	
						22,500
A's month-dollars						$135,500

B

INVESTMENTS				
Jan. 1	$ 6,000 × 12 months	=	$ 72,000	
Aug. 1	4,000 × 5 months	=	20,000	
Oct. 1	2,000 × 3 months	=	6,000	
				$ 98,000
WITHDRAWALS				
July 1	$ 1,000 × 6 months	=	$ 6,000	
Dec. 1	1,000 × 1 month	=	1,000	
				7,000
B's month-dollars				$ 91,000

The distribution of the profits is the same as in the preceding example.

Problems

1. *A*, *B*, and *C* began business January 1. Their accounts for the year appear as follows:

A

		Jan. 1	$ 7,500
		July 1	2,500

B

May 1	$4,000	Jan. 1	$10,000

C

Oct. 1	$7,000	Jan. 1	$10,000
		Aug. 1	3,000

Their profits for the year were $3,310. Determine the share of each partner, if profits were divided on the basis of average investment.

2. Ames and Brown engaged in the hardware business, and at the end of the first year their books showed a profit of $2,357 01. They had agreed to share profits and losses equally, after allowing 6% interest on average investment. Their investments and withdrawals for the year were:

Ames

July 1	$ 500	Jan. 1	$3,000
		Sept. 15	1,500

Brown

Sept. 15	$1,000	Jan. 1	$2,500
		July 1	250

Determine the net capital of each partner at the end of the year.

3. C. H. John and C. B. Arthur formed a partnership. John invested $15,000, but four months later withdrew $3,000. Arthur invested $10,000, and eight months later withdrew $2,000. Interest at 6% was to be credited on average investment; the remainder of the profits was to be distributed in proportion to original investments. The first year's profits, before interest adjustment, were $2,500. What was the net capital of each partner at the beginning of the second year?

Liquidation of partnership. Because of the nature of the association, a partnership must necessarily be terminated on or

before the death of any one of the partners. It is not necessary to discuss here the various causes of dissolution, but only the problems met with at the time of settlement. The purpose of the formation of a partnership is the making of profits, and the division of losses is governed by the same general rule as the division of profits. Profits should be credited and losses should be charged before any division of assets is made. If this rule were not followed, an unfair distribution of capital would result.

When dissolution is accompanied by liquidation, each of the partners has an equal obligation to share in the work. But since it does not usually require the time of all the partners, any one of the partners, or an outsider, may liquidate the business.

In liquidation, profits or losses must first be divided in the profit or loss ratio, and the remaining capital should then be shared by the partners in the capital ratio.

In insolvency, partners must share losses in the profit and loss ratio, and not in the capital ratio. This may at times result in a deficit in capital for some one or more of the partners. Each partner with a deficit should contribute to the firm the amount of his deficit. But if he is totally unable to pay into the firm any portion of his deficit, the remaining partners must bear this loss in the profit and loss ratio.

The governing profit and loss ratio, when a partner is unable to pay, should be stated in fractions, of which the numerators are the profit-and-loss-sharing per cents of the partners with credit balances, and the denominators are the sums thereof. It is evident that it is incorrect to compute the test loss division by multiplying the loss by the profit and loss per cents, since the full amount of the test loss would not be distributed.

Methods. Liquidation may be accomplished in two ways:

(1) All the assets may be converted , all the liabilities paid, the profits or losses distributed, and all the capital divided at one time.

(2) A periodic distribution of the capital may be made before all the assets are converted.

Total distribution. The first method of liquidation does not involve any very difficult calculations.

Example

From the following figures, show the amount of capital distributed to each partner at dissolution:

	A	*B*	*C*
Capital balances before conversion of assets	$10,000	$6,000	$4,000
Profit ratio	40%	40%	20%
Assets converted into cash	$30,000		
Liabilities to be paid	14,000		

Solution

Assets		*Liabilities*		*Net Assets*
$30,000	−	$14,000	=	$16,000

Total Investment, $20,000, less Net Assets, $16,000 = Loss, $4,000

	A (*40%*)	*B* (*40%*)	*C* (*20%*)	*Total* (*100%*)
Capital balances before conversion of assets.	$10,000	$6,000	$4,000	$20,000
Distribution of loss........................	1,600	1,600	800	4,000
Balances..................................	$ 8,400	$4,400	$3,200	$16,000
Cash distributed..........................	8,400	4,400	3,200	16,000

Periodic distribution. Periodic distribution may result from either of two causes:

(1) The desire of the partners to reduce the capital of the firm, or to completely dissolve the firm, even though it is still solvent.

(2) Forced liquidation.

As the assets are converted into cash, and the debts are paid, the balance of cash should be distributed periodically to the partners. This should be done in such a way as to reduce the accounts to the profit and loss ratio existing among the partners. The distribution is made on the assumption that all book assets may be a total loss until converted into cash.

The following example illustrates the adjustment of capital ratios to profit and loss ratios.

Example

From the following data, show the periodic distribution of the cash collected:

	A	*B*	*C*	
Capital balances before conversion of assets..........	$10,000	$6,000	$4,000	
Profit ratio..............	40%	40%	20%	
First period:				
Net loss				$ 1,000
Cash collected ...				9,000
Assets unrealized ..				10,000
Second period:				
Net loss..				1,000
Cash collected .. .				5,000
Assets unrealized . .				4,000
Third period:				
Cash collected				2,000
All other assets uncollectible.				

Solution

	A	B	C	Total
1. Capital balances before conversion of assets	$10,000	$6,000	$4,000	$20,000
2. Distribution of loss	400	400	200	1,000
3. Balance after distribution of loss	$ 9,600	$5,600	$3,800	$19,000
For the purpose of making a test, it will be assumed that the unrealized assets will never be realized.				
4. Test loss in profit and loss ratio	(4,000)	(4,000)	(2,000)	(10,000)
5. After the test loss has been deducted, the remaining amounts will show the proper distribution of the cash balance (3-4)	5,600	1,600	1,800	9,000
6. Balance at the end of the first period (3-5)	$ 4,000	$4,000	$2,000	$10,000
7. Net loss for second period	400	400	200	1,000
8. Balance after distribution of loss	$ 3,600	$3,600	$1,800	$ 9,000
The balances of the accounts are now in the profit and loss ratio.				
9. Distribution of cash	2,000	2,000	1,000	5,000
10. Balance at end of second period	$ 1,600	$1,600	$ 800	$ 4,000
11. Net loss for third period	800	800	400	2,000
12. Balance after distribution of loss	$ 800	$ 800	$ 400	$ 2,000
13. Cash distribution	800	800	400	2,000

The following example illustrates the adjustment of capital to the profit and loss ratio, where a deficiency of one partner is involved.

Example

Show how each period's cash should be distributed in the following:

	A	B	C	
Capital balances before conversion of assets	$10,000	$8,000	$2,000	
Profits to be shared equally.				
First period:				
Net loss				$1,500
Cash to be distributed				8,000
Second period:				
Net loss				1,500
Cash to be distributed				3,000
Third period:				
Remaining assets sold for				4,000

Solution

	A	B	C	Total
Capital balances before conversion of assets	$10,000	$8,000	$2,000	$20,000
First period's loss distributed	500	500	500	1,500
Balance after distribution of loss	$ 9,500	$7,500	$1,500	$18,500
Test loss of amount of the remaining assets	(3,500)	(3,500)	(3,500)	(10,500)
It will be observed from the test loss that *C*'s possible loss is $2,000 greater than his capital. If the test loss should become				

an actual loss, C will owe the firm \$2,000, and if C should be unable to pay in this \$2,000, A and B would be required to bear this additional loss. To provide against this contingency, a further test loss charge of \$2,000 is made against A and B	(1,000)	(1,000)	2,000	
When the sum of the two test losses, \$4,500 (\$3,500 + \$1,000), is deducted from A's investment of \$9,500, it can be seen that A should receive \$5,000; it can also be seen that the sum of B's test losses deducted from his investment gives the amount of cash which is payable to him.				
Cash distribution	\$ 5,000	\$3,000		\$ 8,000
Balance of capital undistributed	\$ 4,500	\$4,500	\$1,500	\$10,500
Second period's loss distributed	500	500	500	1,500
Balance after distribution of loss	\$ 4,000	\$4,000	\$1,000	\$ 9,000
Test loss of unrealized assets	(2,000)	(2,000)	(2,000)	(6,000)
What applied above applies again here. C's account shows a possible loss, and the amount must be distributed as a test loss to be taken up by the other partners.				
Test loss for C's account	(500)	(500)	1,000	
Cash distribution	1,500	1,500		3,000
Balance undistributed	\$ 2,500	\$2,500	\$1,000	\$ 6,000
Third period's net loss	667	667	666	2,000
Cash on hand	\$ 1,833	\$1,833	\$ 334	\$ 4,000
Cash distributed	1,833	1,833	334	4,000

The following example illustrates a return of investments and loans of partners which is complicated by the accounting principle that "loans must be paid before capital is returned to partners."

Example

Partners	*Capital Accounts*	*Loan Accounts*	*Profit Ratio*
A	\$33,000	\$10,500	40%
B	28,500	10,000	30%
C	18,000	21,000	20%
D	10,500	18,500	10%
	\$90,000	\$60,000	100%

The partners have decided upon a dissolution, and after paying all their liabilities, they have:

Cash	\$ 30,000
Other assets	110,000
Net loss	10,000

How should the cash be distributed among the partners, assuming that no member of the firm has private property with which to repay a capital account that has been reduced by losses to a debit balance?

Solution

	A	*B*	*C*	*D*	*Total*
Profit and loss ratio.... ..	40%	30%	20%	10%	100%
Capital balances before conversion of assets..............	$33,000	$28,500	$18,000	$10,500	$90,000
Loss distributed	4,000	3,000	2,000	1,000	10,000
Balance of capital....	$29,000	$25,500	$16,000	$ 9,500	$80,000
Test loss of assets	(44,000)	(33,000)	(22,000)	(11,000)	(110,000)
Possible deficiency of capital	$15,000	$ 7,500	$ 6,000	$ 1,500	$30,000

By applying the possible loss of unrealized assets against capital accounts, it is found that the possible loss is greater than the capital invested.

In practice, as long as the firm has assets and owes each partner money on a loan account, a partner will generally not pay cash into the firm, since the firm would have to pay it back immediately. The problem states that none of the members of the firm has money other than that invested in the firm. As the shortages are debts due the firm, and as the partners have loan accounts, these loan accounts will undoubtedly be used as a set-off. Therefore, the shortages will be deducted as follows:

	A	*B*	*C*	*D*	*Total*
Loans by partners	$10,500	$10,000	$21,000	$18,500	$60,000
Less possible shortages	15,000	7,500	6,000	1,500	30,000
Each partner's standing in the business after distribution of the test loss	($4,500)	$ 2,500	$15,000	$17,000	$30,000

As *A*'s loan is not enough to take up the possible shortage, and as the problem states that *A* has no other property, it is necessary to distribute *A*'s test shortage to the other partners.

	A	*B*	*C*	*D*	*Total*
Ratio of distribution............		$\frac{30}{60}$	$\frac{20}{60}$	$\frac{10}{60}$	
Balances after distribution of test loss........................	($4,500)	$2,500	$15,000	$17,000	$30,000
A's test shortage distributed .	4,500	(2,250)	(1,500)	(750)	
Balances	0	$ 250	$13,500	$16,250	$30,000
Cash distribution..............	0	250	13,500	16,250	30,000

The accounts of the partnership now stand:

Net assets........................		$110,000	
A's capital.....			$ 29,000
B's capital			25,500
C's capital..........			16,000
D's capital..			9,500
A's loan..........................			10,500
B's loan............	$10,000		
Less payment..................	250		9,750
C's loan..........................	$21,000		
Less payment.......	13,500		7,500
D's loan..........................	$18,500		
Less payment..................	16,250		2,250
		$110,000	$110,000

Problems

1. *A*, *B*, and *C* decided to dissolve partnership. On the basis of the following facts, show the proper distribution for each period:

	A	*B*	*C*	
Capital	$8,000	$4,000	$6,000	
Ratios	$44\frac{4}{9}$%	$22\frac{2}{9}$%	$33\frac{1}{3}$%	
First period:				
Net loss				$4,000
Assets uncollected				9,000
Cash				5,000
Final period:				
Assets converted				7,000
Losses				2,000

2. Show the periodic cash distribution based on the following facts:

	Capital	*Loans*	*Ratio*	
X	$22,000	$2,000	50%	
Y	8,000	3,000	$33\frac{1}{3}$%	
Z	4,000	1,000	$16\frac{2}{3}$%	
First period:				
Cash				$ 4,000
Assets uncollected				36,000
Second period:				
Cash				10,000
Assets uncollected				24,000
Loss				2,000
Third period:				
Cash				21,000
Loss				3,000

C. P. A. Problems

1.* The capital of a partnership is contributed as follows:

A	$90,000
B	45,000
C	15,000

The partnership agreement provides for profit sharing in the following ratios:

A	50%
B	30%
C	20%

The partners' salaries are as follows:

A	$5,000
B	3,000
C	2,000

At the end of the first year's business, *C* dies. The books are closed, and the net assets of the business are shown to be $152,500. *A* and *B* liquidate the affairs of the partnership, and distribute the surplus as follows:

First distribution	$42,410.20
Second distribution	74,622.30
Final distribution	31,967.50

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* C. P. A., Maryland.

Prepare a statement of the partners' accounts, showing how the distribution of assets should be made and how the losses should be apportioned.

2.* *A*, *B*, *C*, and *D* enter into partnership with a capital of $100,000. *A* invests $40,000; *B*, $30,000; *C*, $20,000; and *D*, $10,000. They are to share profits or losses in the following proportions: *A*, 35%; *B*, 28%; *C*, 22%; and *D*, 15%. They are also to receive stipulated salaries chargeable to the business.

At the end of six months, there is a loss of $8,000, and meantime the partners have drawn against prospective profits as follows: *A*, $400; *B*, $600; *C*, $600; and *D*, $400.

They dissolve partnership, and agree to distribute the proceeds of firm assets monthly as realized. *C* and *D* enter other businesses, and *A* and *B* remain to wind up the firm's affairs, it being stipulated that from all moneys collected and paid over to *C* and *D*, a commission of 5% be deducted and divided equally between *A* and *B* for their services in liquidating the partnership.

The realization and liquidation lasts four months, and the transactions are as follows:

	Assets Realized	*Liabilities Liquidated*	*Expenses and Losses on Realization, Exclusive of Commissions*
First month	$ 30,190	$ 7,900	$ 400
Second month	50,300	6,100	750
Third month	20,010	3,800	340
Fourth month	9,500	2,200	110
	$110,000	$20,000	$1,600

Prepare partners' accounts, showing the amount payable monthly to each partner.

3.† *A*, *B*, *C*, and *D* formed a personal-service partnership, the clientele of the firm being personal clients of the respective partners.

All fees received and all expenses were pooled by the firm, and the partnership agreement stated that the net earnings for the year were to be shared as follows:

A	$33\frac{1}{3}$%
B	40%
C	$16\frac{2}{3}$%
D	10%

On August 31, as a result of a dispute, a supplementary agreement covering the remainder of the year was made between the partners. This agreement provided that the distribution of net earnings was to be made on the basis of the above percentages, except that in the distribution of the net earnings for the last four months of the year, so far as *C* and *D* were concerned, a net earning was to be assumed on the basis of payment by the clients of *A* and *B* of gross fees of $175,000 and $250,000, respectively, instead of the amounts actually received from those clients.

The deficiency in *A*'s gross fees was to be charged to him, and the excess in *B*'s gross fees credited to him.

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* C. P. A., New York.

† American Institute Examination.

No adjustment for expenses was to be applicable to either the deficiency or the excess.

The net income from January 1 to August 31 was $75,000.

From September 1 to December 31, the following gross fees were received:

From clients of *A*	$110,000
From clients of *B*	290,000
From clients of *C*	15,000
From clients of *D*	25,000

The operating expenses for the last four months were $55,000.

Determine the total net income of each partner for the year, taking into account the supplementary agreement.

4.* On January 1, 19—, Adams, Burk, and Cline became partners in the operation of a dry goods business in Scranton, Pa.

At December 31 of the same year, the trial balance of the partnership, before any adjustments were made, was as follows:

Adams, capital		$ 50,000
Burk, capital		30,000
Cline, capital		20,000
Inventory of merchandise, January 1	$125,000	
Accounts receivable, customers	75,000	
Accounts receivable, employees	3,000	
Cash	6,000	
Notes payable		60,000
Accounts payable		15,000
Sales		500,000
Purchases, including freight	323,000	
Salaries and store expenses	125,000	
Bad debts written off	2,500	
Interest paid on notes payable	6,000	
Salary to Mr. Adams	2,500	
Salary to Mr. Burk	4,000	
Salary to Mr. Cline	3,000	
	$675,000	$675,000

Prepare a balance sheet as of December 31, a profit and loss statement for the year ended the same date, and a statement of the partners' accounts after the following adjustments have been made:

Interest to be credited on partners' capital at 6% per annum.

Mr. Adams owns the store, which the partnership occupies under an agreement providing for an annual rent of $10,000 payable in monthly installments in advance. No rent has been paid during the year. The year's rent should therefore be credited to Adams, together with $325 interest on unpaid monthly installments.

Of the interest paid on notes payable, $2,000 applies to the period subsequent to December 31; accrued taxes, $1,000; accrued wages, $1,500. A reserve of $1,500 is required to cover possible losses from doubtful accounts.

Ten per cent of the profits, if any, after the foregoing adjustments have been made, is to be credited to "Bonuses to department managers and salesmen."

* C. P. A., Pennsylvania.

The remaining profits or losses are to be apportioned to the partners as follows:

Mr. Adams	40%
Mr. Burk	33⅓%
Mr. Cline	26⅔%

5.* A partnership composed of two members divides its profits equally, after all items of income and expense for each calendar year have been determined. One of the items of income is interest on partners' withdrawals, which is calculated and charged to each partner at the end of the year. By agreement, the interest calculation is made on the partners' average monthly balances as shown by the books. Partner *A*'s account for the calendar year 19—, before interest is charged to him, is found to be as follows:

	Debits	*Credits*
January 1, 19—, Balance	$ 1,080 21	$........
January account	6,000 00	550.00
February account	2,500 00	550 00
March account	3,052 74	550.00
April account	13,009 81	9,550 00
May account	5.45	550 00
June account	1,154 20	550 00
July account	1,500 00	550 00
August account	1,500 00	550 00
September account	500 00	550 00
October account	1,000 00	4,050 00
November account	1,014 10	550.00
December account	1,000 00	550 00

Show a statement of the interest which partner *A* should be charged at December 31, 19—; simple interest, 6% per annum.

6.† *A* and *B* are in partnership. *A* receives two-thirds and *B* one-third of the profits. On November 30, 1933, the Profit and Loss account (after interest on capital has been charged at 5%), shows a profit of $6,000. On December 1, 1932, the start of the year under audit, *A* had a capital of $10,000.00 in the business, and during the year he has drawn out $4,500.00. *B* on the same date had a capital of $8,000.00, and during the year has drawn out $1,000.00.

Make up the two capital accounts as they should appear on November 30, 1933.

7.† *A*, *B*, and *C* formed a partnership. *A* agreed to furnish $5,000, *B* and *C* each $3,500. *A* was to manage the business, and was to receive one-half of the profits; *B* and *C* were each to receive one-quarter. *A* supplied merchandise valued at $4,250, but no additional cash. *B* turned over to *A*, as manager, $4,500 cash, and *C* turned over $2,750. The business was conducted by *A* for some time, but exact books were not kept. While manager, *A* purchased additional merchandise amounting in all to $37,500, and made sales amounting to $50,000. The cash received and paid out for the partnership was not kept separate from *A*'s personal cash. *B* took over the management to straighten out the affairs. He found accounts receivable amounting to $10,000. Of these he collected $2,250. The remaining merchandise he sold for $250. These

* C. P. A., North Carolina.
† C. P. A., Indiana.

receipts he deposited to the firm's credit in the bank. The balance of accounts receivable proved worthless. The outstanding accounts payable amounted to $1,000, of which $750 had been incurred in purchasing merchandise, while $250 represented expenses. *B* paid these accounts.

A presented receipted claims, showing that during his management he had paid other expenses of $1,200. By mutual agreement, *B* was held to be entitled to $50 on account of interest on excess capital contributed, and *A* and *C* were each charged $37.50 for shortage of contributed capital.

(*a*) Prepare the Trading and Profit and Loss accounts and the accounts of each of the partners, including the final adjustments to be made at the close of the partnership.

(*b*) Show how the above final adjustments would be modified if *A* proved to have no assets or obligations other than those of the partnership.

8.* *A* and *B*, who are partners in a trading firm, decide to admit *C* as from January 1, 1934.

They make an agreement with *C*, as follows:

C is unable to contribute any tangible assets as his capital investment, but agrees to allow his share of the profits to be credited to his capital account until he shall have one-fifth interest. *C* is to share profits and losses to the extent of one-fifth.

C is to receive a salary of $30,000 per annum, payable monthly, in addition to his share of the profits.

The balance sheet of *A* and *B* at December 31, 1933, is as follows:

Assets		*Liabilities*		
Cash	$ 1,500	Accounts Payable		$ 8,000
Accounts Receivable	10,000	Capital Accounts:		
Merchandise	7,500	*A*	$10,000	
Furniture and Fixtures	1,500	*B*	5,000	
Goodwill	2,500			15,000
	$23,000			$23,000

During the six months ended June 30, 1934, the business has sustained unusual losses, and it is decided to dissolve the partnership.

The balance sheet at that date is as follows:

Assets		*Liabilities*		
Cash	$ 500	Accounts Payable		$12,500
Accounts Receivable	12,500	Capital Accounts:		
Merchandise	5,000	*A*	$10,000	
Furniture and Fixtures	1,500	*B*	5,000	
Goodwill	2,500			15,000
Deficit: Being loss on trading for 6 mos	5,500			
	$27,500			$27,500

Accounts receivable were sold for $9,000, the buyer assuming all responsibility for collection and loss, if any.

Merchandise realized $6,500, and furniture and fixtures $500.

You are asked to make an examination of the accounts from January 1, and to prepare statements showing the realization of assets, the adjustment of the partnership accounts, and the distribution of funds.

* American Institute Examination.

In your examination, you find that C has not drawn his salary for four months, and that B has advanced to the partnership \$2,500 as a temporary loan. You find that these liabilities are included in the sum of \$12,500 shown as accounts payable.

C is ascertained to have no assets.

9.* A, B, and C were in partnership, A's capital being \$90,000, B's \$50,000 and C's \$50,000. By agreement, the profits were to be shared in the following ratio: A, 60%; B, 15%; C, 25%. During the year, C withdrew \$10,000. Net losses on the business during the year were \$15,000, and it was decided to liquidate. It is uncertain how much the assets will ultimately yield, although none of them is known to be bad. The partners therefore mutually agree that as the assets are liquidated, distribution of cash on hand shall be made monthly in such a manner as to avoid, so far as feasible, the possibility of one partner's being paid cash which he might later have to repay to another. Collections are made as follows: May, \$15,000, June, \$13,000; July, \$52,000. After this no more can be collected. Show the partners' accounts, indicating how the cash is distributed in each installment; the essential feature in the distribution is the observance of the agreement given above.

10.† Brown, Green, and Black engage in a soliciting business under an agreement that Brown is to receive a salary of \$200 per month, Green a salary of \$150 per month, and Black a salary of \$100 per month; that the earnings are to be determined at any time at the request of any partner; and that the profits of the business are to be divided on the basis of the amount of business secured by each.

The partnership is in business nine (9) months, and the business record for that period is as follows:

Brown's business	\$4,500.00
Green's business	2,800.00
Black's business	3,000 00

Net profits of the business amount to \$5,026.50.

The partners then decide to rescind the agreement as to salaries, and to divide the profits on the basis of business secured individually, treating all salaries drawn as advances.

Drawings:

Brown	\$1,600 00
Green	1,200 00
Black	900 00

You find that the following errors have occurred during the nine months:

Office furniture charged to expense		\$ 65 00
Accts. rec. (Green's business) worthless		210 00
Cash advanced by Black—credited to his account as business secured		400 00
Items not paid nor entered in the books:		
Brown's salary	\$200.00	
Green's salary	150.00	
Advertising	27 50	
Clerk hire	130.00	

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* American Institute Examination.

† C. P. A., Indiana.

Telephone	6 00
Rent	50 00
Stationery and supplies—exp	15 00

Show the journal entries necessary to readjust the accounts. Make up a statement of the Profit and Loss account, showing all corrections and the distribution of the profits.

11.* Brown and Green entered into a joint venture.

On May 1, 19—, they purchased 5,000 tons of coal in Philadelphia at $4 per ton, f.o.b., for which they gave notes on May 10 for one-half at 3 months and for the other half at 6 months. The coal was shipped to Mexico City on May 15, the freight, and so forth, amounting to $5,000.

A joint banking account was opened on May 10, each party contributing $6,000.

The freight was paid by check on May 20, and on May 25 a check was drawn for $1,000 for charges at Mexico City.

The coal was sold at $7 per ton, and the proceeds used to purchase a cargo of timber, which was shipped to Philadelphia. Freight and other charges thereon, amounting to $3,750, were paid by check June 30.

During July, four-fifths of the timber was sold for $32,000. This amount was received and paid into the joint account August 2.

In order to close the transaction, Brown agreed to take over the remaining one-fifth at cost price, including freight and charges, and he paid a check for this into the joint account August 10.

The first note fell due and was paid August 13, and on the same day the other note was paid under discount at the rate of 4% per annum.

Prepare accounts showing the results of the foregoing transactions; disregard interest on capital contributions.

* American Institute Examination.

CHAPTER 21

Goodwill

Definition. Goodwill is an intangible asset, and may be defined in general terms as the value of any benefits or advantages which may accrue to a business from its being soundly established, bearing a good reputation, having a favorable location, and so forth. It results in the earning of a higher rate of net income than that of less fortunate concerns in the same line of business.

Basis of valuation. When two or more businesses are consolidated or merged, the payment made for each business depends upon:

(1) The value of the net assets of each business.
(2) The earning power of each business.

A committee should be formed, consisting of members from each of the businesses being consolidated or merged (proprietorship, firm, or corporation); this committee should have the assistance of an appraiser and an accountant in the preparation of a report dealing with the net assets and the earning power.

The report should contain a balance sheet of each business, stating the values at which it is proposed to take over the assets, and stating the liabilities to be assumed.

The value of the fixed assets and of the inventory should be determined by the appraiser. The accountant, after making an audit, should submit the other balance sheet items.

Earning power determined from profit and loss statements. The following points should receive consideration when earning power is being determined from profit and loss statements:

(1) *Number of years included.* The value of goodwill depends to some extent on whether profits have been uniform year after year, or have steadily increased or decreased, or have fluctuated from year to year. Therefore, in order to show the trend of profits, it is necessary to have profit and loss statements for several years. A statement of average profits is insufficient, as it does not show the trend.

(2) *Adjustments to correct profits.* Adjustments may be necessary to correct errors, such as:

(*a*) Wrong classification of capital and revenue expenditures.
(*b*) Omission of provision for depreciation, bad debts, and so forth.
(*c*) Inadequate provision for repairs.
(*d*) Anticipation of profits on consignments and sales for future delivery.

(3) *Uniformity of methods.*

(*a*) If the methods of computing the manufacturing costs are not uniform, the cost statements should be revised and put on a uniform basis.
(*b*) The depreciation charges should be analyzed as to method and rate. If different methods and rates have been used, adjustments should be made so that the charges will have been calculated on a uniform basis.
(*c*) There may be a wide difference in the management salaries paid by the consolidating companies for the same services. The salaries should be adjusted. In a single proprietorship or partnership, salaries may not have been paid or credited; in that case they should be included at an arbitrary figure.
(*d*) If, in a partnership, interest on capital has been charged as an expense, the entries should be reversed and the item of interest on capital thus eliminated.

(4) *Eliminations.* Eliminations may have to be made for extraordinary and non-operating profits or losses.

Methods of valuing goodwill. Goodwill may be valued on the basis of:

(1) An appraisal of goodwill.
(2) A number of years' purchase price of the net profits.
(3) A number of years' purchase price of excess profits over interest on net assets.

Capitalization of profits in excess of interest on net assets is usually calculated as follows:

Net assets	$100,000.00
Profits	10,000 00
Interest on net assets @ 6%	6,000 00
Excess of profits over interest	4,000 00
Excess capitalized at 20% (4,000 ÷ .20)	20,000 00

Case illustrations. The following four cases of goodwill valuation, taken from reports of consolidations, show how goodwill has been valued in practice.

Case 1. The goodwill of the consolidating units was fixed at the sum of the profits for the two preceding years, plus an additional 10%.

Case 2. The goodwill was based on the total profits for the five years preceding, less five years' interest on the net worth.

Case 3. The goodwill was the average annual earnings for the four years preceding consolidation, less the following deductions:

(*a*) Profits on favorable contracts about to expire.
(*b*) $100,000 for the estimated value of services rendered by the retiring president.
(*c*) 6% interest on actual capital invested.

The remainder was capitalized on a 10% basis.

Case 4. From the net profits of each company the following items were deducted:

(*a*) 7% on capital actually employed.
(*b*) 1½% on sales.
(*c*) 2% depreciation on brick buildings.
(*d*) 4% depreciation on frame buildings.
(*e*) 8% depreciation on machinery.

The remainder was capitalized at 20%, or 5 times the amount of such earnings in excess of 7% on capital and other deductions agreed upon.

Valuation by appraisal. There is no particular problem in the calculation of the value of goodwill by appraisal. It may be appraised by a disinterested party; or, more often, it is the amount on which the vendor and the vendee agree. They usually appraise the net assets, and agree that the purchase price shall be a certain amount in excess of the value of the net assets. This excess is the payment for goodwill.

Valuation by number of years' purchase price of net profits. The goodwill may be estimated at so many years' purchase price of the net or gross profits of any one year, or at so many years' purchase price of the average profits of a number of years.

Example

The consideration of the sale of a business, as agreed to between the parties, is four years' purchase price of the average profits for the preceding three years, plus the net value of the assets.

Net value of assets		$100,000
Profits of preceding three years:		
1st year	$20,000	
2nd year	15,000	
3rd year	28,000	

What is the selling price of the business?

Solution

Net value of assets		$100,000
Profits of preceding three years:		
1st year	$20,000	
2nd year	15,000	
3rd year	28,000	
	$63,000	
$63,000 ÷ 3 = $21,000, average profits for three years.		
$21,000 × 4 (goodwill)		84,000
Selling price		$184,000

Valuation on basis of excess of profits over interest on net assets. The value of goodwill is calculated under this method by, first, deducting from the average profits a fair return of interest on the capital invested, and, second, by multiplying the remainder of the profits, or the excess, by an agreed number of years' purchase price.

Example

A agrees to buy a certain business, and to pay for it in cash. He agrees to give dollar for dollar of the value of the net assets, plus a six years' purchase price of the excess of the profits over the interest on capital at 6%. Net assets are valued at $100,000, and average profits are $18,000. What is the purchase price of the business, including goodwill?

Solution

Net assets		$100,000
Profits, average	$18,000	
$100,000 × .06	6,000	
Excess profits	$12,000	
$12,000 × 6 (goodwill)		72,000
Purchase price		$172,000

It would be more favorable to the seller to determine the value by using a higher rate of interest and capitalizing the excess profits at this rate; thus:

Net assets		$100,000
Profits, average	$18,000	
$100,000 × .08	8,000	
Excess profits	$10,000	
$10,000 ÷ .08 (goodwill)		125,000
Purchase price		$225,000

$225,000 − $172,000 = $53,000, advantage to the seller.

In the foregoing example, the goodwill represents the capitalization of that portion of the profits which is not attributable to the net tangible assets. The rate to be used depends largely on the kind of business under consideration. In some lines of business the per cent may be as low as 6% or 8%; in others it may be 10%; and in still others 15%, or even 20%.

Basis of stock allotment. Since most phases of the calculation of the value of goodwill are found in consolidations, an example of consolidation is given. The matter of stock allotment is included, because when an agreement has been reached as to the valuation of the assets and as to the earning power of each of the businesses, the next question to decide is the method of making payment.

The following three typical methods will be presented:

(1) Payment entirely in common stock.

(2) Payment in preferred stock for the net assets; payment in common stock for the goodwill.

(3) Payment in bonds for the fixed assets, or for an agreed percentage thereof; payment in preferred stock for the balance of the net assets; payment in common stock for the goodwill.

In the allotment of securities, the fundamental rule is to distribute them in such a manner that, if the income of the consolidation is the same as the combined income of the several businesses, each of the old businesses, or the former owners or stockholders thereof, will receive the same net income as before the consolidation.

To illustrate how this principle would operate under each of the three methods outlined, assume that three companies are to be consolidated on the basis of the following statements:

	A	*B*	*C*	*Total*
Net assets	$40,000	$60,000	$120,000	$220,000
Average earnings.... .	4,000	12,000	20,000	36,000
Rate of income on net assets	10%	20%	16⅔%	

Common stock only. When only common stock is to be issued, it must be issued in the ratio of the net earnings if the income of the consolidation is to be distributed in the ratio in which the companies contributed earnings. To determine the amount of stock which is to be issued, capitalize the earnings by dividing the income of each company by a rate of income agreed upon. Thus, if it is agreed that the rate be 10%, the distribution of common stock is made as follows:

	A	*B*	*C*	*Total*
Stock to be issued:				
A: $4,000 ÷ .10......	$40,000			
B: $12,000 ÷ .10.....		$120,000		
C: $20,000 ÷ .10.....			$200,000	
Total................				$360,000
Less net assets transferred.............	40,000	60,000	120,000	220,000
Goodwill	0	$ 60,000	$ 80,000	$140,000

Ten per cent was chosen as the basic rate, because it was the lowest rate earned by any one of the three companies.

If the profits of the consolidation amount to $36,000, it will be possible to pay a 10% dividend, which would be distributed as follows:

A: 10% of $40,000	$ 4,000
B: 10% of $120,000	12,000
C: 10% of $200,000	20,000
	$36,000

This is an equitable division, so far as profits are concerned. However, it is objectionable because it gives each old company an interest in the assets which is proportionate to the profits earned before the consolidation, instead of an interest proportionate to the assets contributed. This might work a hardship in case of liquidation.

	Net Assets	*Goodwill*	*Total*	*Fraction*
A....................	$ 40,000	0	$ 40,000	$\frac{40}{360}$
B....................	60,000	$ 60,000	120,000	$\frac{120}{360}$
C....................	120,000	80,000	200,000	$\frac{200}{360}$
	$220,000	$140,000	$360,000	

Assume that after a number of years it is decided to liquidate the consolidated company, and that in the meantime, all of the profits have been paid out as dividends. The goodwill has no realizable value, so there is $220,000 to be distributed as follows:

Former stockholders of *A*: $\frac{40}{360}$ of $220,000	$ 24,444.45
Former stockholders of *B*: $\frac{120}{360}$ of 220,000	73,333 33
Former stockholders of *C*: $\frac{200}{360}$ of 220,000 .	122,222 22
	$220,000 00

The former stockholders of *A* would lose, and the former stockholders of *B* and *C* would profit.

Former Stockholders of Company	*Assets Contributed*	*Liquidating Dividend*	*Gain*	*Loss*
A..........	$ 40,000	$ 24,444.45		$15,555.55
B..........	60,000	73,333 33	$13,333 33	
C..........	120,000	122,222 22	2,222 22	
	$220,000	$220,000 00	$15,555 55	$15,555 55

Preferred stock for net assets. In order to avoid giving an advantage to one or more companies at the expense of the others, it is advisable to issue preferred stock for the net assets, and common stock for the goodwill. The goodwill should be allotted to the several companies in the ratio of the excess of the profits contributed over the dividends on the preferred stock.

Assume that in the above illustration 6% stock, preferred as to assets, is to be issued for the net assets, and that common stock is to be issued for the goodwill.

	A	*B*	*C*	*Total*
Earnings	$4,000	$12,000	$20,000	$36,000
Less dividends on preferred stock:				
A: 6% of $ 40,000	2,400			
B. 6% of 60,000		3,600		
C: 6% of 120,000			7,200	
Excess earnings	$1,600	$ 8,400	$12,800	

Common stock should be issued in the ratio of the excess earnings. If five years' purchase of the excess profits were agreed upon, the distribution of stock would be:

	A	*B*	*C*	*Total*
Preferred stock	$40,000	$60,000	$120,000	$220,000
Common stock	8,000	42,000	64,000	114,000

Assuming profits of $36,000 as before, the distribution of dividends would be:

Profits	$36,000
Preferred dividends: 6% of $220,000	13,200
Balance available for common stock dividends	$22,800

Then, $22,800 ÷ $114,000 = 20%, the rate per cent which could be paid on the common stock.

	A	*B*	*C*
Preferred dividends:			
6% of $ 40,000	$2,400		
6% of 60,000		$ 3,600	
6% of 120,000			$ 7,200
Common dividends:			
20% of $ 8,000	1,600		
20% of 42,000		8,400	
20% of 64,000			12,800
Total dividends	$4,000	$12,000	$20,000

These dividends are in each case equal to the profits contributed to the consolidation by the several companies.

It is important to note that goodwill should be based on the profits contributed minus the profits to be returned as preferred dividends, and not on the total profits.

Bonds, preferred stock, and common stock. If bonds are issued for a percentage of the net assets, preferred stock for the remaining net assets, and common stock for the goodwill, the goodwill should be based on the profits turned in minus the bond interest and the preferred dividends.

Assume that 5% bonds are to be issued for 80% of the net assets, 6% preferred stock for the remaining net assets, and common stock for the goodwill, which is to be computed by capitalizing at 20% the earnings of each company in excess of bond interest and preferred dividends to be paid to former stockholders. The issues of the three classes of securities would be computed as follows:

			A	*B*	*C*	*Total*
Bonds:						
A: 80% of $ 40,000			$32,000			
B: 80% of 60,000				$48,000		
C: 80% of 120,000					$96,000	
Total bonds						$176,000
Preferred Stock:						
A: 20% of $ 40,000			8,000			
B: 20% of 60,000				12,000		
C: 20% of 120,000					24,000	
Total preferred stock						44,000
Common Stock:						
A: Earnings		$ 4,000				
B: Bond interest	$1,600					
Pfd. dividend	480	2,080				
Excess		$ 1,920				
$1,920 ÷ .20			$ 9,600			
B: Earnings		$12,000				
Bond interest	$2,400					
Pfd. dividend	720	3,120				
Excess		$ 8,880				
$8,880 ÷ .20				$44,400		
C: Earnings		$20,000				
Bond interest	$4,800					
Pfd. dividend	1,440	6,240				
Excess		$13,760				
$13,760 ÷ .20					$68,800	
Total common stock						$122,800

With profits of $36,000 before allowance for bond interest and preferred dividends, the former stockholders would receive interest and dividends as follows:

	A	*B*	*C*	*Total*
Bond interest:				
5% of $32,000	$1,600			
5% of 48,000		$ 2,400		
5% of 96,000			$ 4,800	
Total				$ 8,800
Preferred dividends:				
6% of $ 8,000	480			
6% of 12,000		720		
6% of 24,000			1,440	
Total				2,640

Common dividends:				
20% of $ 9,600	$1,920			
20% of 44,400		$ 8,880		
20% of 68,800			$13,760	
Total				$24,560
Total distribution	$4,000	$12,000	$20,000	$36,000

Conclusion. The illustrations given are merely indicative of the principles to be borne in mind in the distribution of stock and other securities; they cannot be accepted as procedures to be invariably followed, for several reasons.

First, in the illustrations, the profits of the consolidation are assumed to be the same as the combined profits of the separate companies before they were consolidated. However, consolidations are usually made with the object of increasing profits; hence the question is raised as to how the additional profits should be divided. Should the preferred stock be participating or non-participating?

Second, the question of control involves the matter of the voting rights of the several classes of stock.

These and other considerations would tend to cause modifications in the methods described, but the illustrations serve to indicate the basic principles which must be followed in security allotment in order that the stockholders of the several consolidating companies may preserve their interests in the assets and earnings of the consolidation.

Problems

1. *A*, *B*, and *C* are about to consolidate. The following data are presented:

	A	*B*	*C*	*Total*
Net Assets	$250,000	$150,000	$600,000	$1,000,000
Average Profits	50,000	15,000	150,000	215,000
Interest Rate				10%
Profit Rate	20%	10%	25%	

Prepare tabulations showing the stock distribution:

(*a*) Preferred stock for the net assets, and common stock for the goodwill.

(*b*) Show possible disadvantage of issuing only common stock.

2. Using the data in Problem 1, show the security allotment if 5% bonds are issued for 80% of the net assets, 6% preferred stock for the remainder, and common stock for the goodwill, which is to be based on excess earnings capitalized at 15%.

3. *A*, *B*, and *C* call upon you to draw up plans for their consolidation. They submit the following information:

Assets	*A*	*B*	*C*
Plants	$350,000	$200,000	$180,000
Materials	100,000	20,000	20,000
Accounts Receivable	80,000	60,000	40,000
Cash	20,000	10,000	10,000

Liabilities	*A*	*B*	*C*
Accounts Payable...............	$ 70,000	$ 30,000	$ 20,000
Capital........................	350,000	200,000	100,000
Surplus........................	130,000	60,000	130,000
Average income.................	30,000	35,000	40,000

Upon your recommendation, the consolidation will issue: (*a*) 6% bonds for the fixed assets; (*b*) 7% preferred stock for the remaining net assets; (*c*) common stock for the goodwill, which is to be based on excess profits capitalized at 10%.

Assuming that the consolidation will have net profits amounting to $105,000, prepare statements showing the allotment of securities and the distribution of profits.

4.* The net worth and profits of three companies are as follows:

X	
Capital....................................	$100,000
Profits....................................	50,000
Y	
Capital....................................	200,000
Profits....................................	50,000
Z	
Capital....................................	250,000
Profits....................................	50,000

(*a*) Give your theory of how a consolidation should be made.

(*b*) Show the respective interests of *X*, of *Y*, and of *Z* in the consolidated company, using a factor of 6% to represent the normal value of money.

5.† *A* has agreed to sell to *B* the goodwill of the X. Y. Company on the basis of three years' profits of the business, which are to be determined by you, on sound principles of accounting and as accurately as possible, from the following statement handed you by *A*. You are required to compute the value of the goodwill, but are not expected to take into account any considerations except those presented by the statement.

Credits	*1st Year*	*2nd Year*	*3rd Year*
Sales (selling prices substantially uniform throughout period)..........	$638,400	$602,500	$ 564,000
Estimated value of construction work performed and charged to property	110,000	77,600	154,000
Appreciation of real estate upon revaluation by experts..............		80,000	
Profit on sale of Bethlehem Steel Co. stock..........................			85,000
Inventory at end of period:			
Production material at cost	72,000	103,100	106,600
Finished goods at selling prices.....	76,500	114,000	150,000
	$896,900	$977,200	$1,059,600

* C. P. A., Michigan.

† American Institute Examination.

Debits			
Production materials purchased......	$233,000	$252,400	$ 220,300
Production labor............	50,850	61,400	60,900
Production expense (including depreciation)......................	66,750	69,300	70,300
Selling expenses....................	52,500	55,650	62,800
Interest	96,000	94,000	98,500
Cost of construction work	74,600	49,000	86,000
Inventory at beginning of period:			
Production material at cost........	51,400	72,000	103,100
Finished goods at selling prices.....	54,900	76,500	114,000
	$680,000	$730,250	$ 815,900
Balance, being profit claimed by *A*...	$216,900	$246,950	$ 243,700

6.* In the preceding problem, does the basis used for arriving at the value of the goodwill—three years' profits—appear to you to be reasonable in view of the facts disclosed to you? If not, what advice would you offer upon the question if *A* or *B* were your client?

7.* *A* and *B* are partners in business, and have the following statement:

Store	$15,000	Accounts Payable...........	$10,000
Accounts Receivable	12,000	Bills Payable	5,000
Cash	9,000	*A*'s Capital	30,000
Furniture and Fixtures	2,800	*B*'s Capital.................	35,000
Merchandise	37,000		
Miscellaneous Equipment . .	4,200		
	$80,000		$80,000

C is admitted as a special partner, under the following arrangement: *C* is to contribute $30,000, and is to be entitled to one-third of the profits for 1 year. Before the contribution is made, the following changes are to be made in the books: store to be marked down 5%; allowance for doubtful accounts to be created, amounting to 2%; merchandise to be revalued at $35,000; furniture and fixtures to be revalued at $2,500. At the end of the year, the goodwill is to be fixed at 3 times the net profits for the year in excess of $20,000, this goodwill to be set up on the books and the corresponding credit to be to *A* and *B* equally. *A*, *B*, and *C* are each to draw $3,000 in cash, and the remaining profits are to be carried to their capital accounts.

During the year, the following transactions took place:

Merchandise bought on credit	$240,000
Cash purchases..................	25,000
Cash sales...................	125,000
Sales on credit.............	175,000
Accounts payable paid (face, $245,000; discount, 2%)...	240,100
Accounts receivable collected (face, $170,000; all net except $50,000, on which 2% was allowed)	169,000
Buying expenses, paid cash	1,500
Selling expenses, paid cash	21,000
Delivery expenses, paid cash	9,000
Management expenses, paid cash	4,500
Miscellaneous expenses, paid cash	3,000
Interest on notes payable, paid cash.................	250

* American Institute Examination.

The partners each withdrew $3,000 cash, as agreed.

When the books were closed for the purpose of determining the profits and goodwill, the following were agreed upon:

Value of merchandise on hand	$60,000
Depreciation on store	285
Additional allowance for doubtful debts	165
Furniture and fixtures written down	200

The goodwill having been estimated and duly entered, *C* then contributes enough cash to make his capital account equal one-third of the total capital.

Prepare statements showing how the accounts are to be adjusted, and prepare the balance sheet after the final adjustment.

CHAPTER 22

Business Finance

Stock rights. Corporations, in undertaking to secure additional capital, not infrequently offer additional stock to their stockholders at a price below the prevailing market quotation of the outstanding shares.

This privilege of subscribing has value as long as the market price of the old stock remains higher than the offering price of the new stock; and if the stockholders prefer not to exercise the rights, they may sell them in the market for whatever they will bring.

Example

A corporation has a capital stock of $100,000, divided into 1,000 common shares of $100 par value. The entire amount is outstanding, and the market quotation is $150. Finding that $50,000 additional capital is needed, the directors decide to offer to the stockholders 500 shares of new common stock at $125. They accordingly announce on August 1 that stockholders of record as of September 1 will have the privilege of subscribing for the new issue in the proportion of one share of the new stock for every two shares of the old stock held on the latter date. The subscriptions are payable on or before October 1 following, and transferable warrants for the rights are to be issued as soon as practicable after September 1. What is the value of a right?

Explanation. According to the conditions of this offer, every holder of two of the old shares at the close of business on September 1 will be entitled to subscribe to one of the new shares. He will therefore come into possession of two "rights," as that term is used on the New York Stock Exchange, or one right for every old share held. (On some stock exchanges the term right indicates the privilege of subscribing to one share of the new issue.)

Trading in the rights will begin following the declaration of the directors on August 1, and will continue until October 1. Until the warrants are in the hands of the stockholders—during the period from August 1 to September 1—the trading will be on a "when issued" basis; that is, delivery and payment for the rights will be made when the warrants are available. During this time the stock will sell "rights-on"; that is, the market value of the shares will include the value of the rights.

With the delivery of the warrants on September 1, the stock will sell "ex-rights," its price no longer including the value of the rights. With the issuance of the warrants, and until October 1, trading in the rights will be for immediate delivery and payment; that is, delivery and payment the day after the sale is made.

Should a holder of two shares exercise his privilege of subscription, he would own:

2 shares @ $150		$300
1 share @ 125		125
3 shares @ 141.67		$425

It will be noticed that the difference between the market price of the old stock and the average price of the three shares is $8.33, the value of a right; also that the difference between the market price of the old stock and the offering price of the new stock is $25.00, or three times the value of a right.

Therefore, the following formula may be used:

Formula

$$\frac{\text{Market price} - \text{Offering price}}{\text{Number of rights to purchase 1 share} + 1} = \text{Value of a right.}$$

Substitution

$$\frac{150 - 125}{2 + 1} = \frac{25}{3}, \text{ or } \$8.33.$$

During the second period—that is, while the stock is quoted ex-rights—the value of the rights may be ascertained as follows:

Formula

$$\frac{\text{Market price} - \text{Offering price}}{\text{Number of rights to purchase 1 share}} = \text{Value of a right.}$$

It will be found that the market price of the rights during the second period will tend to coincide with this value. Any appreciable difference opens an opportunity for a profit.

The foregoing discussion and example apply only to values on the market. The profit or loss resulting from the sale of rights, and the profit or loss from the sale of stock acquired by the exercise of rights, are governed by Section 29.22(*a*)-8, Regulations 111.

Sale of stock and rights, federal income tax. Ordinarily, a stockholder derives no taxable income from the receipt of rights to subscribe for stock, nor from the exercise of such rights, but if he sells the rights instead of exercising them, he may derive taxable income, or sustain a loss.

The following rule is stated in Sec. 29.22(*a*)-8, Regulations 111.

"(1) If the shareholder does not exercise, but sells, his rights to subscribe, the cost or other basis, properly adjusted, of the stock in respect of which the rights are acquired shall be apportioned between the rights and the stock in proportion to the respective values thereof at the time the rights are issued, and the basis for determining gain or loss from the sale of a right on one hand or a share of stock on the other will be the quotient of the cost or other basis, properly adjusted, assigned to the rights or the stock, divided, as the case may be, by the number of rights acquired or by the number of shares held."

Example

A purchased 100 shares of stock at $125.00 a share, and in the following year the corporation increased its capital by 20%. *A*, therefore, received 100 rights, entitling him to subscribe to 20 additional shares of stock; the subscription price was $100.00 a share. Assume that at the time that the rights were issued the stock had a fair market value of $120.00 a share, and that the rights had a fair market value of $3.00 each. If, instead of subscribing for the additional shares, *A* sold the rights at $4.00 each, his taxable gain would be computed as follows:

100 shares @ $125.00....	$12,500.00, cost of stock in respect of which rights were issued
100 shares @ $120.00. .	$12,000.00, market value of old stock
100 rights @ $3.00 . .	$300.00, market value of rights
$\frac{12,000}{12,300}$ of 12,500........	$12,195.12, cost of old stock apportioned to such stock after issuance of rights
$\frac{300}{12,300}$ of 12,500........	$304.88, cost of old stock apportioned to rights
100 rights @ $4.00 . . .	$400.00, sales price of rights
$400.00 – $304.88	$95.12, profit on sale of rights

For the purpose of determining the gain or loss from the subsequent sale of the stock in respect of which the rights were issued, the adjusted cost of each share is $121.95—that is, $12,195.12 ÷ 100.

Rule 2 of Sec. 29.22(*a*)-8, Regulations 111, states:

"(2) If the shareholder exercises his rights to subscribe, the basis for determining gain or loss from a subsequent sale of a share of the stock in respect of which the rights were acquired shall be determined as in paragraph (1). The basis for determining gain or loss from a subsequent sale of a share of the stock obtained through exercising the rights shall be determined by dividing the part of the cost or other basis, properly adjusted, of the old shares assigned to the rights, plus the subscription price of the new shares, by the number of new shares acquired."

Example

A purchased 100 shares of stock at $125.00 a share, and in the following year the corporation increased its capital by 20%. *A*, therefore, received 100 rights entitling him to subscribe to 20 additional shares of stock; the subscription price was $100.00 a share. Assume that at the time that the rights were issued the stock had a fair market value of $120.00 a share, and that the rights had a fair market value of $3.00 each. *A* exercised his rights to subscribe, and later sold for $140.00 a share 10 of the 20 shares thus acquired. The profit is computed as follows:

Cost of old stock apportioned to rights in accordance with the computation in the example under Rule 1...	$ 304 88
Subscription price of 20 shares at $100.00 a share......	2,000.00
Basis for determining gain or loss from sale of shares acquired by exercise of rights......................	$2,304.88

$2,304.88 ÷ 20 = $115.24, basis for determining gain or loss from sale of each share of stock acquired by exercise of rights.

Proceeds of sale:	
10 shares @ $140.00	$1,400 00
Cost of stock sold:	
10 shares @ $115.24	1,152 40
Profit	$ 247 60

The basis for determining the gain or loss from the subsequent sale of the remaining 10 shares of stock acquired on subscription is $115.24 a share; and the basis for determining the gain or loss on the stock in respect of which the rights were issued is $121.95 a share—that is, $12,195.12 ÷ 100, as in the example under Rule 1.

Problems

1. A company has a capital stock of $1,000,000, divided into 10,000 common shares of $100 par value. The entire amount is outstanding, and the market quotation is $150 a share. Finding that $500,000 of additional capital is needed, the directors decide to offer to the stockholders 5,000 shares of new common stock at par. What is the approximate market value of a right if each stockholder may subscribe for one share of new stock for every two shares of the old stock held?

2. A corporation offered, at $100 a share, one share of its new stock for each six shares held. The stock was selling at $185 a share when the offer was announced. What was the approximate market value of a right?

3. *W* owned 100 shares of Purity Baking Common that cost him $13,400. Later, he received rights to subscribe to additional stock, but since he did not care to increase his investment, he sold the rights at $3\frac{5}{8}$ less commission, receiving therefor $357.30. At the date when the stock was quoted ex-rights, the average market values were:

Stock	$124\frac{1}{2}$
Rights	$3\frac{5}{8}$

(*a*) What was *W*'s loss on the sale of the rights?
(*b*) What was the carrying value of the stock?

4. Smith owned 100 shares of common stock in the W. Corporation, which offered rights to subscribe to new common stock at $100 a share, the basis of the offering being one share for each five shares held. The average market values on the date when the stock sold ex-rights were:

Stock	150.50
Rights	11.8125

Smith later sold his rights at $14.50.

(*a*) If Smith paid $120 a share for the original 100 shares, what is his profit on the sale of the rights?
(*b*) What is the carrying value of the 100 shares?

Working capital. One of the most difficult problems for anyone entering a new business is to know how much money will be required to finance the enterprise until the receipts will equal or

exceed the disbursements. While this is strictly a question of finance, the accountant is often called upon to deal with it.

Example

A manufacturer gives you the following data, and requests that you estimate the amount of working capital required to finance the making and selling of an article:

Selling price, each	$100	
Cost to make, each	60	
Selling expenses, each	20	
Overhead, each	10	
Net profit, each	10	
Sales, first month	50	articles
" second month	100	"
" third month	150	"
" fourth month	200	"
" each month thereafter	200	"

All the sales are installment sales, the payments being $10 per month. Assume arbitrarily that the complete cost of $90 on each article is incurred at the time that the sale is made.

What will be the largest amount of capital required, and in which month will it be required?

Solution

Months	*Total Costs Each Month*	*Total Receipts Each Month*	*Deficiency Each Month*	*Working Capital Required*
First	$ 4,500	$ 500	$ 4,000	$ 4,000
Second	9,000	1,500	7,500	11,500
Third	13,500	3,000	10,500	22,000
Fourth	18,000	5,000	13,000	35,000
Fifth	18,000	7,000	11,000	46,000
Sixth	18,000	9,000	9,000	55,000
Seventh	18,000	11,000	7,000	62,000
Eighth	18,000	13,000	5,000	67,000
Ninth	18,000	15,000	3,000	70,000
Tenth	18,000	17,000	1,000	71,000
Eleventh	18,000	18,500	500†	70,500
Twelfth	18,000	19,500	1,500†	69,000

† Receipts from collections are more than the costs for the month.

The above table shows in the last column the amount of working capital required to finance the business by months. The greatest amount required is found to be $71,000 in the tenth month. Thereafter, the collections are greater than the costs.

Problems

1. A company is about to be formed for the purpose of manufacturing a specialty. After careful investigation, the following estimates have been made:

Selling price, each	$75
Cost to make, each	35
Selling and administration expense	14
Net profit	26

Sales:		
First month	30	machines
Second "	70	"
Third "	180	"
Fourth "	200	"
Each month thereafter	225	"

The terms of payment are $15 down, and $5 per month. What is the greatest amount of working capital that will be required, and in which month will this amount be needed?

2. The X. Company plans to sell on the installment basis, direct from factory to consumer. Their product is a specialty retailing at $100, payable $10 with order and balance in nine equal installments.

Cost to manufacture:	
Material	40%
Labor	35%
Burden	25%
Selling expense	15% of sales
Administration expense	4% of sales
Other expense	1% of sales

Labor cost is expected to increase $14\frac{2}{7}$%, which will decrease the profit 35%.

Estimated sales:		
First month	50	machines
Second "	100	"
Third "	150	"
Fourth "	200	"
Each month thereafter	200	"

Assuming that all the expenses of a sale are paid during the month in which the sale is made, prepare a schedule showing the essential facts, and the amount of working capital needed monthly.

3.* On the basis of the following facts, determine, by months, the cash requirements of an installment dealer for the first year's operations:

1. Cost of article $50 00
2. Sales price 90 00
3. Selling expense 15 00
4. Overhead 15 00
5. Profit 10.00
6. Sales for the first month were 100 articles
7. Sales for the second month were 200 articles
8. Sales for subsequent months were 300 articles per month
9. Merchandise paid for on the month following the sale
10. Expenses paid during the month of sale
11. Payments are received at the rate of $10 down and $10 per month; assume that no irregularities are experienced

4.† The A. B. Company acquired the right to sell musical instruments in a given territory. They request you:

* C. P. A., Wisconsin.
† C. P. A., Pennsylvania.

(a) To state how much capital will be required to carry on the business during the first year.

(b) To demonstrate by computation how your estimates would work out during the first six months.

Assume that the sales for the first year will total \$180,000 from the sale of instruments, and \$24,000 from service work (respectively, \$15,000 and \$2,000 monthly). The overhead and direct selling costs are estimated at \$30,000 for the year. This amount includes all expenses except the cost of instruments sold and parts used in service, the latter being estimated at \$12,000.

The instruments are purchased on 60 days' credit, at a discount of 30% from the price at which they are sold to the customer. Twenty per cent of the instruments are sold for cash, and 80% on lease contracts. When they are sold on lease contracts, 25% is required as a down payment, and the balance in 12 months. All payments are made to the A. B. Company. The leases are discounted at the bank, the charges by the bank being added to the price charged the customer. The service charges are billed and payable in 30 days.

Cumulative voting. Cumulative voting is a method whereby each shareholder is entitled to cast a number of votes equal to the product of the number of shares which he holds and the number of directors to be elected. The shareholder may cast all his votes for any one or more of the directors to be elected, or may distribute his votes in any way that he desires. Thus, the minority stockholders, by combining their votes, may elect a representative on the board of directors.

Example

A corporation has an outstanding capital stock of \$100,000, composed of 1,000 shares of common stock with a par value of \$100 each. The stockholders are to elect seven directors at the annual meeting. Calculate the least number of shares required to elect three of the directors, provided that the cumulative method of voting is used.

Formula	*Substitution*
$\frac{a \times c}{b + 1} + 1 = x.$	$\frac{1{,}000 \times 3}{7 + 1} + 1 = 376.$

Explanation.

a = Number of shares outstanding
b = Number of directors to be elected
c = Number of directors minority desires to elect
x = Required number of shares
$1{,}000 \times 7$ = 7,000, total votes of 1,000 shares
376×7 = 2,632, total votes of 376 shares
$2{,}632 \div 3$ = 877, the number of votes each of the three directors would receive if the holders of the 376 shares of stock cast all their votes for three directors
$7{,}000 - 2{,}632$ = 4,368, balance of votes
$4{,}368 \div 5$ = 873, the largest number of votes the remaining stockholders could cast for five directors

By the same method of calculation, it will be found that the owners of 375 shares would have 875 votes for each of three directors, while the remaining stockholders would have 875 votes for each of five directors. This would give a tie vote, and neither side could elect the desired number.

Problems

1. In a corporation which uses the cumulative method of voting, how many of the seven directors can you safely seek to elect, if you own 1,501 out of the 4,000 voting shares?

2. Seven directors are to be elected by the X. Company, which has a voting capital of 5,000 shares. How many shares are necessary to elect four directors under the cumulative plan?

Book value of shares of stock. A corporation is owned by the stockholders, whose evidences of ownership are shares of stock. The book value of a share of stock is equal to the quotient of the net worth divided by the number of shares of stock outstanding.

Formula

$$\frac{\text{Assets} - \text{Liabilities}}{\text{Number of shares}} = \text{Book value per share}$$

Example

A corporation has assets of $340,000 and liabilities of $120,000. It has a capital stock of $200,000. If the shares have a par value of $100 each, what is their book value?

Solution

$340,000 − $120,000	$220,000 00
$220,000 ÷ 2,000 shares	110.00

Problems

1. Find the book value of a share of stock in each of the following companies

	Assets	*Liabilities*	*Capital*	*Par Value of Shares*
(*a*)	$350,829.75	$134,082.47	$100,000	$100
(*b*)	$575,850.00	$190,260.75	25,000 shares	No Par
(*c*)	$1,322,080 35	$110,809.20	$1,000,000	$50

2. A company has assets of $385,915.28 and liabilities of $158,910.75. If there are 5,000 shares outstanding, each with a par value of $50, what is the book value of each share?

Profits distribution. The distribution of profits in the partnership type of business organization was discussed under the subject of partnership (Chapter 20). The distribution of the profits of corporations is illustrated in the following examples.

Example 1

At the end of a certain year, a corporation had outstanding 2,250 shares of common stock, par value $100. A 2% dividend was declared. Net earnings

were $53,320.84. What was the amount of the dividend, and what amount of the year's profits, after the declaration of the dividend, remained in surplus?

Solution

2,250 shares @ $100	$225,000, capital stock
2% of $225,000	$4,500, dividend
$53,320.84 − $4,500	$48,820.84, credit to surplus

Example 2

A company had the following number of outstanding shares, each with a par value of $50: common, 858,860 shares; 6% cumulative preferred, 291,047 shares; 5% non-cumulative and non-participating preferred, 28,849 shares.

The company declared a 6% dividend on the common stock. What amount of profits was distributed to each class of stock? What amount of profits was necessary to cover the dividends?

Solution

	Capital	*Dividends*
858,860 shares common @ $50	$42,943,000 00	
6% of $42,943,000		$2,576,580 00
291,047 shares 6% cumulative preferred @ $50	14,552,350.00	
6% of $14,552,350		873,141.00
28,849 shares 5% non-cumulative preferred @ $50	1,442,450.00	
5% of $1,442,450		72,122.50
Profits necessary to cover dividends		$3,521,843.50

Problems

1. Calculate the amount of dividends on the following stocks; each class has a par value of $100:

182,260 shares common @ 6%;
57,633 shares 7% cumulative preferred.

2. A company earned $2,850,460.03. It paid from this 8% dividends on $12,379,850 preferred stock outstanding. If the preferred stock was non-participating, what per cent was earned for the common stock, of which there was $10,600,000 outstanding? Par value in each case was $50 a share.

3. The outstanding stock of a corporation consists of:

Preferred *A* stock, without par value, series 1, cumulative dividends $7 per share	10,000 shares
Participating preference stock, without par value, cumulative dividends $8 per share	16,301 shares
Preferred *B* stock, without par value, non-cumulative dividends $3.50 per share	6,659 shares
Common stock without par value	198,145 shares

The participating preference stock entitles the holder to receive, among other things, participating dividends equal share for share to any dividends paid from time to time on the common stock.

If, in the course of a year, 50¢ is paid on each share of common stock, what will be the total amount of dividends paid?

CHAPTER 23

Public Finance and Taxation

Governmental functions. Governmental functions are divided among three major classes of governmental units—the federal government, the forty-eight state governments, and thousands of local government bodies, including counties, towns, townships, cities, villages, and such special units as drainage, levee, irrigation, and park districts.

Purposes of taxes. Federal government taxes are used to pay the army and navy, the salaries of governmental personnel, pensions, and other goverment expenses such as education, highways, economic development, social welfare, and so forth. Expenses of the federal government in 1943 amounted to more than 78 billion dollars; most of this was for national defense, which cost 72 billions. The budget included a billion dollars for aid to agriculture and almost 2 billions for interest on the public debt.

State government taxes are used to pay the salaries of state government personnel, the support of schools, universities, and asylums, and for sundry other state expenses. The largest single item in state expenditures for the 48 states during the year 1942 was that for operation and maintenance, which in that year amounted to $4,083,877,000; of this amount, $1,030,117,000 was for operation and maintenance of schools.

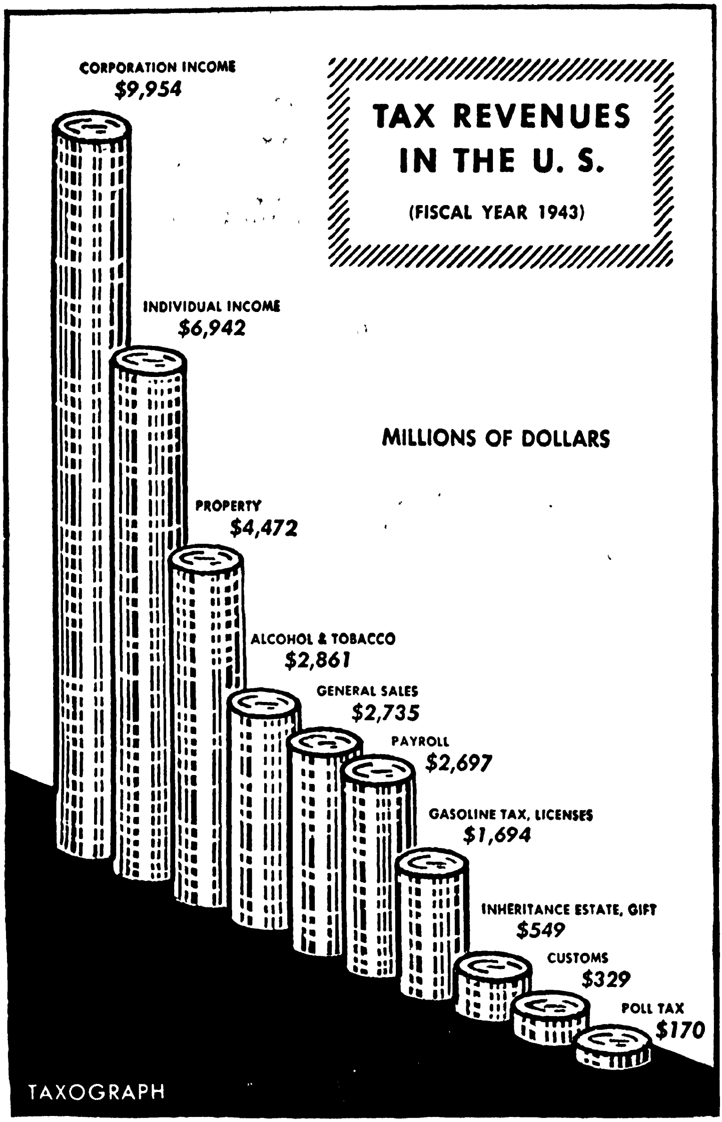
County taxes are used to pay the salaries of county employees, the cost of roads, charities, and miscellaneous other county expenses.

City taxes are used to pay the salaries of city employees, police and fire protection, support of schools, and other city expenses.

Town taxes are used to pay the salaries of town employees, support of schools, and other town expenses.

Taxes levied by special units are for the purpose of paying for and maintaining the special units.

Appropriations. Funds deemed to be necessary for the conduct of government are set up by the respective governing bodies as appropriations. The size of the appropriations may result from modifications in the quantity and/or quality of governmental activities and from changes in commodity prices and wage and



Courtesy of Minnesota Taxpayers' Assn.

salary levels. To meet these appropriations, taxes are levied upon persons and property.

Kinds of taxes. Commodity taxes may be limited in their scope, applying to particular commodities, such as taxes on tobacco, liquors, and so forth; or they may be general, applying to the manufacture and sale of all or most commodities, such as sales taxes, manufacturers' excise taxes, and so forth.

Highway taxes are best known as the motor vehicle tax or license, and the motor vehicle fuel tax known as the gasoline tax.

The general property tax and the special property taxes are the major sources of revenue for state and local governments. The tax rate imposed on the assessed value of a piece of property is a total of a series of separate tax rates imposed by local and state governments. Real property is listed and valued by the assessor, while personal property is usually listed by the taxpayer, who sets his own value on the articles he lists, although the assessor may change the values if investigation proves them to be incorrect.

Taxes on business consist of state bank taxes and state taxes on insurance companies, railroads, and public service enterprises such as telephone and light and power companies. There are also state taxes on business in general in the form of license fees and franchise taxes.

Income taxes are levied on the income of both persons and business. The first income tax was a federal tax, but now most of the states have income tax laws. The excess profits tax (repealed in 1946) is a tax on the excess profits of a corporation and is a federal tax.

Death taxes on transfer of property of a deceased person to his beneficiaries or heirs are levied by the federal government and by most of the state governments. One form is the federal estate tax, another is the state inheritance tax. To prevent distribution of large holdings of property in order to escape estate and inheritance taxes, the gift tax was enacted by the federal government. Several of the states also have gift taxes.

The chart on page 224 summarizes the general sources of tax revenues.

Income, inheritance, estate, and gift taxes are complex and constitute complete studies in themselves; also, rates and regulations are frequently undergoing changes. Therefore, these subjects will not be presented in this text.

Property Tax

Determination of tax rate. The amount of money needed divided by the assessed valuation of the property determines the rate to be levied. For example:

State tax:
Budget..............................\$ 3,750,000
Assessed value of property in state...... 1,250,000,000
3,750,000 ÷ 1,250,000,000 = .30%, State rate

County tax:		
Budget	$ 90,000	
Assessed value of property in county	4,500,000	
90,000 ÷ 4,500,000 =		2 00%, County rate
City tax:		
Budget	75,000	
Assessed value of property in city	2,500,000	
75,000 ÷ 2,500,000 =		3 00%, City rate
		5.30%, Total rate

To find the amount of tax. Multiply the assessed valuation by the rate. Tax rates may be expressed: as so many mills on the dollar; as a certain rate per cent; or as dollars on the thousand.

Example

Property is assessed at $8,500; the tax rate is $40.60 a thousand. Find the tax.

Solution

$ 40 60, rate on each thousand
8 5 , number of thousands assessed
$345.10, the tax.

Problems

1. What is the total tax rate on the following?

State:	
Budget	$ 5,500,000
Assessed valuation	2,500,000,000
County:	
Budget	72,000
Assessed valuation	6,000,000
City:	
Budget	125,000
Assessed valuation	3,750,000
School District:	
Budget	35,000
Assessed valuation	1,750,000

2. In a certain town the tax rate on $1,000 was as follows:

State		$ 1.20
County		30 00
Town:		
Library	$1.24	
Revenue	.91	
Road and Bridge	2 74	
Road Drag	91	
		5.80
School District		1.00
Total rate		....

If the assessed value of X Company's property in this town was $93,960, what was the amount of property tax?

3. If property is assessed at $6,880 and the tax rate is $100.30 a thousand, what is the tax?

4. If the tax rate is $100.30 a thousand, what is the rate per cent? What is this rate in mills on the dollar?

5. A tax rate of $99.20 a thousand is made up of the following rates:

State:	
Debt	$ 7 33
Road, Bridge, and Soldiers' Relief	1.10
School	1.23
Teachers' Retirement	.04
County:	
Revenue	27.09
City:	
Revenue	61.41
School District	1.00
Total	....

Property valued at $75,000 is assessed at $\frac{3}{5}$ of its value. What per cent of the total tax applies to each division?

6. In a certain city taxes are due January 1, but may be paid in two installments, the first on or before May 31 and the second on or before Oct. 31.

If the first half is not paid on or before May 31, the following penalties will attach: During June, 3%; July, 4%; August, 5%; September, 6%; October, 7%.

During November and December, the penalty is 8% computed on any amount unpaid.

The second one-half cannot be paid until the first one-half has been paid.

If the tax rate is $96.50 a thousand, find the total tax paid on the following:

(*a*) Property with assessed value of $35,000, both installments paid on August 10.

(*b*) Property with assessed value of $7,500, the first installment paid July 15 and the second installment paid December 15.

7.* A city with an assessed valuation of $1,000,000 and estimated receipts for current expenses from miscellaneous sources of $50,000 and of $2,000 from sinking fund investments has submitted to you the following budget of expenditures for the year:

Mayor and other Commissioners	$20,000
Water Department	15,000
Bond Interest	5,000
Fire Department	20,000
Police Department	21,000
Health Department	15,000
Retirement of Bonds	10,000
Street Department	18,000
General Government	25,000

What tax levy must be made to provide the necessary revenue?

8.† The assessed valuation of the taxable property of the State of W., as

* C. P. A., North Carolina.

† Adapted from C. P. A. Examination.

determined by the Tax Commission for a certain year, was $4,068,268,534.

What would a citizen whose property was assessed at $507,374 have to pay for each of the following purposes, and what would be the amount of his total tax bill?

Purposes—Total Amount to Be Raised

Interest on Certificates of Indebtedness	$ 199,339.42
Free High Schools	175,000.00
State Graded Schools	200,000.00
Highway Improvements	1,700,000.00
General Purposes	100.00

In addition to the above, the following mill taxes are assessed:

University	$\frac{3}{8}$ mill
Normal Schools	$\frac{1}{6}$ mill
Common Schools	$\frac{7}{10}$ mill

CHAPTER 24

Fundamentals of Algebra

Explanation. The work of an accountant is complicated in many particulars and requires technical calculations. It is extremely difficult to make some of these calculations by arithmetic. On the other hand, if the fundamentals of algebra are understood, the calculations may be made with comparative ease. Only the more common and more useful principles of algebra will be discussed in this chapter.

Symbols and terms. In algebra, the letters of the alphabet are usually employed as symbols to represent numbers.

The following signs have the same meaning in algebra as in arithmetic:

+ is read "plus"
− is read "minus"
× is read "times" or "multiplied by"
÷ is read "divided by"
= is read "equals"

An exponent is a number or symbol written at the right of another number or symbol and a little above it, to show how many times the latter is to be used as a factor and to indicate its power. For example, "25^3" = $25 \times 25 \times 25 = 15{,}625$. When no exponent is indicated, 1 is understood to be the exponent.

An equation is an expression of equality between two numbers.

An axiom is a statement which is admitted to be true without any proof. Algebraic operations make use of the following axioms:

(1) The equality of both sides of an equation is not destroyed by the addition of the same number to both sides.

(2) The equality of both sides of an equation is not destroyed by the subtraction of the same number from both sides.

(3) The equality of both sides of an equation is not destroyed by the multiplication of both sides by the same number.

(4) The equality of both sides of an equation is not destroyed by the division of both sides by the same number.

Positive and negative numbers. A positive or negative state of any concrete magnitude may be expressed without reference to the unit; thus, numbers that are greater than zero are positive, and numbers that are less than zero are negative.

A number which has a "+," or positive, sign prefixed to it is called a positive number; thus, $+5$. If a "$-$" sign precedes the unit, it is called a negative number, and is written thus, -5.

Addition of positive and negative numbers. When two or more positive and negative numbers are combined into a single number, the result is called the sum of the numbers.

Example

The sum of $+6$ and -4 is $+2$
" " " $+4a$ and $-2a$ is $+2a$
" " " $+2$ and -6 is -4
" " " $+3a$ and $-7a$ is $-4a$
" " " -4 and -3 is -7
" " " $-3a$ and $-4a$ is $-7a$

From the foregoing, the following rules may be derived:

(1) To add two numbers or terms of different signs, subtract the smaller number or term from the larger, and prefix the sign of the larger.

(2) To add two negative numbers, add their absolute values and prefix the negative sign.

Example

Add $+4a$, $-3a$, $+6a$. The problem may be restated thus: $(+4a) + (-3a) + (+6a)$.

Solution

$$\begin{array}{r} +\ 4a \\ \underline{+\ 6a} \\ +10a \\ \underline{-\ 3a} \\ +\ 7a \end{array}$$

The addition of positive quantities is made in the same way as in ordinary addition.

The addition of a positive and a negative number is equivalent to deducting the smaller number from the larger, and retaining the sign of the larger.

Problems

Find the sum of each of the following:

1. $+4, -3, +7, -2, +6, +4, -8, -6.$
2. $-4, -8, +6, +5, -4, -3, -2, -7.$
3. $+1, -4, -6, +2, +7, +6, +4, +5.$

The coefficient. The number or letter put before a mathematical quantity, known or unknown, to show how often it is to be taken, is called the coefficient. In adding terms which are multiples

of the same letter, add the coefficients of these terms, and prefix the proper sign. Thus, $+6a$, $+7b$, $-5a$, $+6b$ becomes:

$$\begin{array}{lrr} & +6a & +\ 7b \\ & -5a & +\ 6b \\ \hline \text{Added} & +\ a & +13b \end{array}$$

It is convenient to arrange the terms in columns, so that like terms stand in the same column.

Problems

Find the sum of each of the following:

1. $+4a,\ +3b,\ -4a,\ -2b$.
2. $+6a,\ +4b,\ -6c,\ +3a,\ -4b,\ +4c$.
3. $+4x,\ -4y,\ +4z,\ -3x,\ +3y,\ +2z$.
4. $-5x,\ +3y,\ +4z,\ +4x,\ -4y,\ -4z$.
5. $-x,\ +4x,\ +3x,\ -12x,\ -4x,\ +x$.
6. $+4x,\ +3x,\ -6x,\ +2x,\ -12x,\ +2x$.
7. $+4a,\ +3b,\ +3c,\ -4c,\ +4b,\ -a$.
8. $-a,\ -b,\ +c,\ +b,\ -c,\ +a,\ -c$.
9. $+c,\ +x,\ -z,\ +y,\ -2c,\ +6x,\ -3z$.
10. $+x^2,\ +2x^2,\ -xy,\ -x^2,\ -3x,\ +y$.

Hereafter, when a term is not preceded by a positive or a negative sign, it is to be understood as a positive term.

Parentheses, brackets, and braces. Parentheses, brackets, and braces are used to indicate that the part inclosed is to be employed as a single term or as a single unit.

Since the same rules apply to all signs of aggregation, in the explanation given hereafter, only parentheses will be mentioned.

RULES. (*a*) When a term in parentheses is preceded by a "+," or positive, sign, the parentheses may be removed without any change in the signs of the inclosed terms.

(*b*) If a term in parentheses is preceded by a "−," or negative, sign, when the parentheses are removed it is necessary to change each of the positive and the negative signs of the terms inclosed.

Examples

$$a + b + (c - d) = a + b + c - d$$
$$a + b - (c - d) = a + b - c + d$$
$$(a - b) + (c - d) - (e - f) = a - b + c - d - e + f$$
$$(4a + 3b - c) - (d + 4e - f) = 4a + 3b - c - d - 4e + f$$

Wherever possible, like terms should be combined; as:

$$(3a - 3b + 4c) - (2a - 3c) = a - 3b + 7c$$

If several algebraic expressions are inclosed one within the other by inclosure signs, such as parentheses or brackets, eliminate the innermost pair of inclosure signs first.

Example 1

$$a + [b - (c - d)] = a + [b - c + d]$$
$$= a + b - c + d.$$

Example 2

$$a + b - [-(b + c) + (c - d)] = a + b - [-b - c + c - d]$$
$$= a + b + b + c - c + d$$
$$= a + 2b + d.$$

Problems

Simplify the following by removing all signs of aggregation:

1. $x + y + (x + y) - (2x + 2y)$.
2. $3a + (b - 4c) - (4a - b - c)$.
3. $42 + (37 + 6) - (40 + 20)$.
4. $30 - [20 + (3 + 4) - (4 + 2)]$.
5. $(3x - 4y) - (6x + 2y) + (4x - 3y)$.
6. $-3a + [4b - (6c + 7a) - 5b + 6c]$.
7. $-3b^2 + 4a^2 - (2b^2 - 4a^2) + 4a$.
8. $a + b + c + d - (2a + 2b - 2c + 2c)$.
9. $-x - y - (z - x) - y - z(x + y + z)$.
10. $(8a - 4b) + (3c + d) - (3a + b + c)$.

Subtraction. Subtraction is the process of determining one of two numbers when their sum and one of the numbers are given.

The minuend is the sum of the two numbers.

The subtrahend is the number to be deducted.

The remainder is the required number.

To subtract, change the sign of the subtrahend and add the subtrahend and the minuend.

Example 1

Subtract $8x$ from $4x$.

Solution

$8x$ from $4x = (+4x) - (+8x)$

Removing parentheses: $= 4x - 8x$

$= -4x$

Example 2

Deduct $-8x$ from $-4x$.

Solution

$-8x$ from $-4x = (-4x) - (-8x)$

Removing parentheses: $= -4x + 8x$

$= +4x$

Example 3

Deduct $-8x$ from $4x$.

Solution

$-8x$ from $4x = (+4x) - (-8x)$

Removing parentheses: $= 4x + 8x$

$= +12x$

To subtract algebraic expressions having two or more terms, change the sign of each term of the subtrahend and proceed as in addition.

Example

Subtract $3a - 5c - 3d$ from $7c - 16a - 2d$.

Solution

The changing of the signs of the subtrahend is usually done mentally; thus.

$$\begin{array}{r} -16a + 7c - 2d \\ 3a - 5c - 3d \\ \hline -19a + 12c + d \end{array}$$

Problems

Subtract:

1. $17x$ from $28x$.
2. $13x$ from $3x$.
3. $16a$ from $-18a$.
4. $-6a$ from $18a$.
5. $a + 2a$ from $8a - 3a$.
6. $4a - 3b$ from $3a + 4b$.
7. $16a - 14b + 3c$ from $-13a + 2b - 4c$.
8. $-16a + 4b + 2c$ from $18a - 14b + 3c$.
9. $3cs - 5bz$ from $-6cs + 7bz$.
10. $3p + 4q$ from $4p - 4q$.

Multiplication. When two or more numbers are multiplied, the result is called the product of the numbers.

When two numbers with like signs, either positive or negative, are multiplied, the product is positive.

Examples

$$(+a) \times (+b) = +ab$$
$$(-a) \times (-b) = +ab$$
$$(-4) \times (-4) = +16$$

When two numbers with unlike signs are multiplied, the product is negative.

Examples

$$(+a) \times (-b) = -ab$$
$$(+4) \times (-3) = -12$$

The exponent to be used in the product is equal to the sum of the exponents appearing in the multiplier and the multiplicand.

Examples

$$(a)^1 \times (a)^1 = a^2$$
$$(a^4) \times (-a^3) = -a^7$$
$$(-3a^2) \times (-2a^3) = +6a^5$$

Problems

Multiply the following:

1. a^2 by a^2.
2. a^3 by a^6.
3. $-a^3$ by a^4.
4. ab by ab.
5. a^2 by b^2.
6. $a + b + c$ by 5.
7. $3a + 100$ by 6.
8. $3(a + b)$.
9. $1{,}000 + s - b$ by 6.
10. $(100 - c) - b$ by 4.
11. $[20{,}000 - (2{,}000 + x)]$ by $12\frac{1}{2}$.
12. $[40{,}000 - (T - B - 2{,}000)]$ by 12.

Division. Division is the process of finding one of two numbers when the other number and the product of the two numbers are given. When the dividend and the divisor have like signs, either positive or negative, the quotient is a positive number.

Examples

$$ab \div a = b$$
$$-ab \div -b = a$$
$$-14 \div -7 = 2$$

When the dividend and the divisor have unlike signs, the quotient is a negative number.

Examples

$$-ab \div a = -b$$
$$ab \div -b = -a$$
$$-10 \div 5 = -2$$
$$10 \div 5 = 2$$

The exponent to be used in the quotient is equal to the difference between the exponent in the dividend and the exponent in the divisor.

Examples

$$a^7 \div a^2 = a^5$$
$$-ab^2 \div b = -ab$$
$$8a^6b^4 \div -2ab^3 = -4a^5b$$
$$20^5 \div 20^2 = 20^3$$

Problems

Divide the following:

1. 63 by 9.
2. -40 by 8.
3. -125 by -5.
4. $48a$ by $12a$.
5. a^2b by a.
6. $9a^2$ by $3a$.
7. $5a^2 - 60b$ by 5.
8. $50a^{10} - 25a^{15}$ by 5.
9. $20{,}000 - (2{,}000 - 5x)$ by 5.
10. $1{,}600 - (200 + 4x)$ by 4.
11. $60a^{28} - 30a^{24} + 15a^{20}$ by $15a^5$.
12. $72a^8 - 18a^{10} - 54a^6$ by $9a^2$.

CHAPTER 25

Equations

Simple equations. An equation is an expression of equality between two magnitudes or operations. The members of the equation are separated by the sign of equality, " = ," which means "is equal to." Either member of an equation may contain numerals, letters, or both. A simple equation is an equation of the first degree, and contains but one unknown.

Example

Simple equation:	$x = 50$
Or:	$10x = 500$

If the same number is added to or subtracted from both sides of an equation, the equality is not destroyed.

Examples

Simple equation:	$x = 50$
Adding 10 to each side:	$x + 10 = 60$
Simple equation:	$10x = 500$
Subtracting 10 from each side:	$10x - 10 = 490$

If both sides of an equation are multiplied by or divided by the same number, the equality is not destroyed.

Example 1

	$3x = 15$
Multiplied by:	5
	$15x = 75$
Simplifying, or dividing by 15:	$x = 5$

Example 2

	$4)20x = 40$
Divided by 4:	$5x = 10$
Simplifying, or dividing by 5:	$x = 2$

Any term of either member of an equation may be transposed from one side to the other by changing the sign of the term, and the

equality of both sides of the equation is not destroyed. This operation is equivalent to either adding the same quantity to, or subtracting the same quantity from, both sides of the equation.

Example 1

Simple equation:	$10x - 10 = 490$
Transposing the "-10" to the right side of the equation, and changing the sign:	$10x = 490 + 10$
Or:	$10x = 500$

Example 2

Simple equation:	$x + 5 = 15$
Transposing:	$x = 15 - 5$
Or:	$x = 10$

Example 3

Solve the equation, $10x - 14 = 6x + 2$.

Solution

The equation:	$10x - 14 = 6x + 2$
Transposing "$6x$" to the left side, and changing the sign to "$-$":	$10x - 6x - 14 = +2$
Transposing "-14" to the right side, and changing the sign:	$10x - 6x = 14 + 2$
Uniting similar terms:	$4x = 16$
Dividing by 4:	$x = 4$

Verification

	Left side		*Right side*
	$10x = 40$		$6x = 24$
	$-14 = -14$		$2 = 2$
Adding:	$\overline{26}$	Adding:	$\overline{26}$

Therefore the two sides of the equation are equal, and $x = 4$.

Problems

In the following equations, solve for the unknown quantities, and check your results:

1. $40x - 20 = 10x + 10$.
2. $10x + 5 = 25$.
3. $8a + 8 = 2a + 32$.
4. $16 + 5a = 8a + 1$.
5. $49b - 4 = 37b + 8$.
6. $2x + 6(4x - 1) = 98$.
7. $39 + 4(a + 6) = 8a - 1$.
8. $2{,}000 - (5x + 500) = 2{,}500$.
9. $8b - 5(4b + 3) = 1 - 4(2b - 7)$.
10. $18y - (10y - 8) = 20y - (6y + 4)$.

Example

How can 90 be divided into two parts in such a way that one part will be four times the other?

Solution

Let:	$x =$ the smaller part
Then:	$4x =$ the larger part
Adding:	$5x = 90$
Dividing each member of the equation by 5:	$x = 18$, the smaller part
	$4x = 72$, the larger part

Example

How many dimes and cents are there in \$2.40, if there are 60 coins in all?

Solution

Let:	$x =$ the number of dimes
Then:	$60 - x =$ the number of cents
Therefore:	$10x =$ the value of the dimes
But:	$60 - x =$ the value of the cents
Simplifying:	$10x + 60 - x = 240$
Transposing:	$9x = 180$
	$x = 20$, or there are 20 dimes
	$60 - x = 40$, or there are 40 cents

Verification

20 dimes	=	\$2.00
40 cents	=	.40
60 coins	=	\$2.40

Example

The P. Q. Company wrote off depreciation on its building, which cost \$50,000, at the rate of 2% of the original cost per annum. This amount was included in General Administration Expense account, and constituted $\frac{1}{10}$ of that account.

The purchases cost two and one-half times the old inventory. The value of the old inventory was twice the amount of the selling expense. The Selling and General Administration Expense accounts each equalled 10% of the sales. The new inventory was valued at an amount equal to the selling expense.

The interest and discount costs were $\frac{1}{5}$ of the selling expense. Set up a profit and loss statement showing the net profit from operations.

Solution

	\$50,000 × 2%	= \$1,000, Depreciation
	\$1,000, Depreciation	= $\frac{1}{10}$ of General Adm. Expense
Multiplying by 10:	\$1,000 × 10 = \$10,000	= General Adm. Expense
	\$10,000	= Selling Expense
	Selling Expense	= $\frac{1}{10}$ of Sales
Multiplying by 10:	\$10,000 × 10	= \$100,000, or Sales
	2 × \$10,000	= \$20,000, or Old Inventory
	$2\frac{1}{2}$ × \$20,000	= \$50,000, Purchases
	$\frac{1}{5}$ of \$10,000	= \$2,000, Interest and Discount

THE P. Q. COMPANY
PROFIT AND LOSS STATEMENT
DECEMBER 31, 19—.

Sales		$100,000
Cost of Sales:		
Old Inventory	$20,000	
Purchases	50,000	
Total	$70,000	
Less New Inventory	10,000	60,000
Gross Profit		$ 40,000
Selling Expense	$10,000	
General Administrative Expense	10,000	20,000
Operating Profit		$ 20,000
Interest and Discount		2,000
Net Profit		$ 18,000

Problems

1. How can 640 be divided into two parts, in such a way that one part will be seven times the other?

2. How many quarters and cents are there in $4.00, if there are 40 coins in all?

3. A sum of money, $10.00, is made up of dimes and quarters, the total number of coins being 88. How many dimes and quarters are there?

4. In a certain dairy, the ice cream contains 14% cream. If the mixture is to be made from coffee cream of 20% butter fat and milk which tests 4%, what portion of each will have to be used in a mixture of 100 lbs.?

5. How many pounds of coffee worth 25¢ per pound must a grocer mix with other coffee worth 42¢ per pound to make a mixture worth 34¢ per pound? The total quantity desired is 50 lbs.

6. Barnes has $6,000 invested in 5% bonds. How much must he invest in 8% stock to make his average net income 6%?

7. An estate of $33,120 was to be divided among the mother, three sons, and three daughters. The mother was to receive four times as much as each son, and each son three times as much as each daughter. How much was each to receive?

8. In the making of candy, a mixture of 75% sugar at 5¢ per pound and 25% corn syrup at 2¢ per pound is used. If the price of sugar advances to 7¢ per pound, what must be the ratio of sugar and corn syrup to be used, if the cost of production is to remain the same?

9. A vinegar manufacturer makes various grades of vinegar, ranging from 50% to 100% pure cider vinegar. How much 100% pure cider vinegar must be mixed with 63% pure to make 100 gallons of 75% pure?

10. How much milk with a butter fat test of 3.5% must be mixed with milk testing 4.65% to fill 75 quart bottles which are to test 3.95%?

Fractions. In many accounting problems it is necessary to use simple fractions in algebraic form, such as "$\frac{1}{10}$ of x" or "$\frac{4}{5}$ times a," which when simplified are written $\frac{x}{10}$ or $\frac{4a}{5}$.

Example

Divide 129 into two parts, in such a way that $\frac{2}{9}$ of one part will equal $\frac{3}{8}$ of the other.

Solution

Let:	$x = \text{one part}$
Then:	$129 - x = \text{the other part}$
The problem states that:	$\frac{2x}{9} = \frac{3}{8}(129 - x)$
The common denominator of the fractions, as seen by inspection, is 72, and by the process of changing the fractions to a common denominator, the result is:	$\frac{16x}{72} = \frac{27}{72}(129 - x)$
Multiplying each side of the equation by 72, to clear the equation of the fractions, the result is:	$16x = 27(129 - x)$
Eliminating the parentheses:	$16x = 3{,}483 - 27x$
Transposing:	$43x = 3{,}483$
Dividing by 43:	$x = 81$
	$129 - x = 48$

Verification

$\frac{2}{9}$ of $81 = 18$
$\frac{3}{8}$ of $48 = 18$

Example

The superintendent of a certain plant was hired under a contract which provided that he was to receive 10% of the net profits of the business as a salary, after his salary had been deducted as an expense. The profits for the year were \$13,200. Compute the superintendent's salary.

Solution

Let:	$s = \text{the amount of salary}$
The problem is stated:	$s = \frac{1}{10}(\$13{,}200 - s)$
Clearing of fractions by multiplication:	$10s = \$13{,}200 - s$
Transposing:	$11s = \$13{,}200$
Dividing by 11:	$s = \$1{,}200$

Verification

Net Profits	\$13,200
Less Salary	1,200
	\$12,000
Dividing by 10	1,200

Clearing of complex fractions. To clear an equation of a complex fraction, multiply the opposite term of the equation by the denominator of the complex fraction, and solve the resulting equation by the usual methods.

Example

Find the value of $(1.06)^{10}$ in the following equation:

$$\frac{1 - \dfrac{1}{(1.06)^{10}}}{.06} = \$7.3600871$$

Solution

STEPS

$$\frac{1 - \dfrac{1}{(1.06)^{10}}}{.06} = \$7.3600871$$

Multiplying right-hand term by .06:	$1 - \dfrac{1}{(1.06)^{10}} = 7.3600871 \times .06$	(*1*)
Clearing:	$1 - \dfrac{1}{(1.06)^{10}} = .4416052$	(*2*)
Transposing and changing signs:	$\dfrac{1}{(1.06)^{10}} = 1 - .4416052$	(*3*)
Clearing:	$\dfrac{1}{(1.06)^{10}} = .5583947$	(*4*)
Changing and dividing:	$(1.06)^{10} = 1 \div .5583947$	(*5*)
Clearing:	$(1.06)^{10} = 1.790847$	(*6*)

Problems

1. One-half of a certain number exceeds one-sixth of the same number by 8. What is the number?

2. Find the value of $(1.05)^{10}$ in the following complex fraction:

$$\frac{1 - \dfrac{1}{(1.05)^{10}}}{.05} = 7.72173.$$

3. Find the value of $(1.03)^{15}$ in the following complex fraction:

$$\frac{(1.03)^{15} - 1}{.03} = 18.598913.$$

4. Find the value of $(1.04)^{20}$ in the following:

$$\frac{1}{\dfrac{1 - \dfrac{1}{(1.04)^{20}}}{.04}} = .0735818$$

5. Find the value of $(1.005)^{48}$ in the following:

$$\frac{1 - \dfrac{1}{(1.005)^{48}}}{.005} = 42.5803178.$$

Simultaneous equations with two or more unknowns. When each of two equations contains two or more unknown quantities, and every equation containing those unknowns may be satisfied by the same set of values for the unknown quantities, the equations are said to be simultaneous.

The value of the unknown quantities in two or more simultaneous equations may sometimes be found by combining the equations into a single equation containing only one unknown quantity.

This combining may be done in several different ways, and is known as elimination.

The method of elimination by addition and subtraction is probably the most simple, and will therefore be the one used here.

It is often necessary to multiply one, or sometimes both, of the equations by a number that will make the terms that contain one of the unknowns in each equation of equal absolute value. Substitute known values where possible.

Add or subtract the resulting equations, and the sum or remainder will be an equation containing one unknown less than the previous equations.

The chief difficulty in the practical application of these rules is the expression of the unknowns in the form of equations. It seems advisable to make a written statement of each condition, equation, or unknown, and also a similar statement of each of the knowns.

After each statement, a symbol or letter should be used to represent each unknown or known. In algebra, the letters "x," "y," and "z" are commonly used, but it seems to be more practical to use the initial letter of the name of the thing whose value is to be found.

Example 1

Carol has five times as much money as Mary. Together they have \$60. How much money has each?

Statement of equations:

M = the number of dollars belonging to Mary
$C = 5M$, or five times as much as belongs to Mary
$M + C = \$60$

Solution

Substitution of $5M$ for C: $M + 5M = 60$
Combining: $6M = 60$
Dividing by 6: $M = 10$
$C = 5M$, or \$50

Example 2

It cost \$98.50 to manufacture and sell a certain article. The cost of the labor was equal to the cost of the material used. The cost of the overhead was \$9.50 more than the expenses. The overhead and expenses totaled \$2.50 more than the material. Find the cost of each item.

Statement of equations:

		Steps
M	= Cost of Material	(*1*)
L	= Cost of Labor	(*2*)
O	= Cost of Overhead	(*3*)
E	= Cost of Expenses	(*4*)
$M + L + O + E$	= \$98.50	(*5*)

Solution

	$M = L$	(*6*)
	$O = E + \$9.50$	(*7*)
	$M = O + E - \$2.50$	(*8*)
	$M = E + \$9.50 + E - \$2.50$	(*9*)
	$M = 2E + \$7.00$	(*10*)
	$M = 2E + \$7.00$ in terms of E	(*11*)
	$L = 2E + \$7.00$ " " " "	(*12*)
	$O = E + \$9.50$ " " " "	(*13*)
	$E = E$ " " " "	(*14*)
Adding (*11*), (*12*), (*13*), and (*14*):	$M + L + O + E = 6E + \$23.50$	(*15*)
Substituting:	$\$98.50 = 6E + \$23.50$	(*16*)
Transposing:	$6E = \$98.50 - \$23.50$	(*17*)
Dividing by 6:	$E = \$12.50$	(*18*)

Substituting in (*11*), (*12*), (*13*), and (*14*), the value of each item may be found.

Example 3

In the following simultaneous equations, solve for the values of a and b:

	Steps
$10a - 6b = 38$	(*1*)
$14a + 8b = 4$	(*2*)

Solution

Multiplying (*1*) by 7:	$70a - 42b = 266$	(*3*)
Multiplying (*2*) by 5:	$\underline{70a + 40b = 20}$	(*4*)
Subtracting (*4*) from (*3*):	$-82b = 246$	(*5*)
Dividing by 82:	$-b = 3$	(*6*)
Or:	$b = -3$	(*7*)
Substituting the value of b in (*1*):	$10a + 18 = 38$	(*8*)
Transposing 18 in step (*8*):	$10a = 20$	(*9*)
Dividing by 10:	$a = 2$	(*10*)

Example 4

The following are the condensed balance sheets of three companies who wish to know their true worth as of December 31:

	Company A	Company B	Company C
Assets, exclusive of intercompany investments	$200,000 00	$500,000 00	$300,000.00
Investment in Company B (50%)	350,000 00		
" " Company C (20%)	250,000 00		
" " Company C (20%)		100,000 00	
" " Company A (10%)			100,000 00
	$800,000 00	$600,000 00	$400,000 00
Capital Stock	$500,000 00	$400,000 00	$300,000 00
Surplus	300,000 00	200,000 00	100,000.00

As each company owns stock in each of the others, there are three unknown quantities. The true worth may be found as follows:

Solution

SUMMARY OF OWNERSHIP

	Net Assets	Company A	Company B	Company C
Company A owns	$200,000 00		50%	20%
Company B owns	500,000 00			20%
Company C owns	300,000 00	10%		

Let: A = Net Worth of Company A
Let: B = Net Worth of Company B
Let: C = Net Worth of Company C

Statement of equations:

$$A = 200{,}000 + \tfrac{1}{2}B + \tfrac{1}{5}C$$
$$B = 500{,}000 + \tfrac{1}{5}C$$
$$C = 300{,}000 + \tfrac{1}{10}A$$

Solution

			Steps
	$A - \frac{1}{2}B - \frac{1}{5}C =$	200,000.00	(1)
	$B - \frac{1}{5}C =$	500,000 00	(2)
	$-\frac{1}{10}A + C =$	300,000 00	(3)
Adding:	$\frac{9}{10}A + \frac{1}{2}B + \frac{3}{5}C =$	1,000,000.00	(4)
Multiplying (4) by 10:	$9A + 5B + 6C =$	10,000,000.00	(5)
Multiplying (2) by 5:	$+5B - C =$	2,500,000 00	(6)
Subtracting (6) from (5):	$9A + 7C =$	7,500,000.00	(7)
Multiplying (3) by 7:	$-\frac{7}{10}A + 7C =$	2,100,000.00	(8)
Subtracting:	$9\frac{7}{10}A =$	5,400,000 00	(9)
Multiplying (9) by 10:	$97A =$	54,000,000 00	(10)
Dividing by 97:	$A =$	556,701 03	(11)
Using (3) and substituting the value of $-\frac{1}{10}A$:	$-55{,}670.10 + C =$	300,000.00	(12)
Transposing:	$C =$	355,670.10	(13)
Using (2) and substituting the value of $-\frac{1}{5}C$:	$B - 71{,}134.02 =$	500,000.00	(14)
Transposing:	$B =$	571,134.02	(15)

Verification

	A	*B*	*C*	*Total*
Assets, exclusive of inter-company investments..	$200,000.00	$500,000.00	$300,000.00	
A owns: $\frac{1}{2}$ of B..........	285,567.01			
$\frac{1}{5}$ of C..........	71,134.02			
Total worth of A........	$556,701.03			$ 556,701.03
B owns: $\frac{1}{5}$ of C..........		71,134.02		
Total worth of B........		$571,134 02		571,134.02
C owns: $\frac{1}{10}$ of A.........			55,670 10	
Total worth of C........			$355,670 10	355,670.10
Total value of the three companies............				$1,483,505 15

Problems

1. For a certain piece of advertising, the total cost of printing, envelopes, and postage is $14.50. The envelopes cost $1.50 more than the postage. The cost of printing is $0.50 more than the combined costs of postage and envelopes. Find the separate costs.

2. At the end of the year, the books of the Blank Company showed a net profit of $12,247.50. The treasurer was to receive $12\frac{1}{2}\%$ of the net profits as a bonus. The treasurer defaulted, and it was found that his account was short $4,847.55. Show: (*a*) the true profit; and (*b*) the treasurer's account balance.

3. A man has $7,000 invested at 5%. How much must he invest at $6\frac{1}{2}\%$ to make his total income equal to 6% on his total investment?

4. In a gallon of a certain kind of paint there are equal parts of pigment and oil. How much oil must be added to a gallon of this paint to make a paint of $\frac{2}{5}$ pigment and $\frac{3}{5}$ oil?

5. A merchant made 25% on his capital the first year, excluding his salary. He withdrew $1,800 for his personal expenses, and had $9,153.13 left. Find the amount of his investment.

6. A certain candy contains 40% corn syrup and 60% sugar. The syrup costs 2¢ per pound and the sugar 5¢ per pound. If sugar advances to 6¢ and the cost of the syrup is unchanged, in what proportions must the ingredients of the mixture be used in order to keep the cost the same?

7. *A* and *B* were partners, and agreed to share profits in proportion to the capital invested. The profits were $2,000. *A* owned a $\frac{2}{5}$ interest plus $400, and his share of the profits was $900. What was the value of the business and the capital of each partner?

8. The audited statements of a company show the following: The cash is $2,400 more than the expenses. The accounts receivable equal twice the amount of cash less $8,000. The cash and the accounts receivable together exceed the expenses by $10,000. Find the amount of cash, accounts receivable, and expenses.

9. The following is the balance sheet of the B. Company:

Assets		*Liabilities*	
Cash	$ 200	Accounts Payable	$ 5,000
Accounts Receivable	2,500	Capital Stock	10,000
Plant and Equipment	18,000	Surplus or Net Profits	13,200
Officers' Accounts:			
President Smith	3,000		
Treasurer Brown	4,500		
	$28,200		$28,200

By agreement, the president was to receive, in lieu of salary, 15% of the net profits, and the treasurer was to receive 10% of the net profits. Both the deductions were to be included in expenses.

The treasurer, who was not bonded, has disappeared, and you are now requested to state the amount of Smith's and of Brown's accounts and also the true amount to be credited to surplus.

10. You are requested to find the value, for consolidated purposes, of the following balance sheets, all as of the same date:

	A	B	C
Assets, other than stock	$400,000	$200,000	$200,000
Stock in B Company	60,000		20,000
Stock in C Company	60,000	20,000	
Deficit		40,000	
	$520,000	$260,000	$220,000
Liabilities	$200,000	$160,000	$ 40,000
Capital Stock	300,000	100,000	100,000
Surplus	20,000		80,000
	$520,000	$260,000	$220,000

The investments in stock were at par and cost, there being neither surplus nor deficit at the date of purchase.

11. Three companies agree to consolidate, and each agrees to accept its pro rata share in the capital stock of the new corporation, D. Corporation D is formed with 500,000 shares of no par value stock.

A

Total Assets	$ 750,000	Total Liabilities	$ 125,000
		Capital Stock	500,000
		Surplus	125,000
	$ 750,000		$ 750,000

B

Total Assets	$1,000,000	Total Liabilities	$ 375,000
		Capital Stock	375,000
		Surplus	250,000
	$1,000,000		$1,000,000

C

Total Assets	$1,750,000	Total Liabilities	$ 550,000
		Capital Stock	1,000,000
		Surplus	200,000
	$1,750,000		$1,750,000

STOCK OWNERSHIP

	A	B	C
A owned		15%	15%
Current at		$50,000	$150,000
B owned	15%		10%
Current at	$75,000		$ 37,500
C owned	5%	5%	
Current at	$25,000	$30,000	

What percentage of the capital stock of Corporation D will each incorporator receive, and what will be the book value of each company's interest in Corporation D?

Arithmetical solution of problems containing unknown quantities. Some accountants prefer to solve problems containing unknown quantities by an arithmetical rather than an algebraic method. The arithmetical method consists of making estimates of the unknown quantities on the basis of quantities which are known. A second test is made, based upon the results of the first estimate. Succeeding tests or approximations are then made until the correct value is ascertained.

A peculiarity of this method is that mistakes made in the computations will always be eliminated, and the final computation will always be correct. An error in computation may necessitate a greater number of approximations, but in the end it will be eliminated in the process of solution.

Example

You are requested to find the value, for consolidated balance sheet purposes, of the following corporations. The separate balance sheets show the value of each corporation as of the same date.

AMES COMPANY

Assets	$200,000	Liabilities	$100,000
Stock of Brown Co. (par)	30,000	Capital Stock	150,000
Stock of Coulter Co. (par)	30,000	Surplus	10,000
	$260,000		$260,000

BROWN COMPANY

Assets	$100,000	Liabilities	$ 80,000
Stock of Coulter Co. (par)	10,000	Capital Stock	50,000
Deficit	20,000		
	$130,000		$130,000

COULTER COMPANY

Assets	$100,000	Liabilities	$ 20,000
Stock of Brown Co. (par)	10,000	Capital Stock	50,000
		Surplus	40,000
	$110,000		$110,000

The holding company purchased the stock of the subsidiaries, paying book value therefor. Book value was par, for the subsidiaries had neither surplus nor deficit at the date of acquisition by the holding company.

REPEATED TRIALS

	First Test	Second Test	Third Test	Fourth Test	Fifth Test	Sixth Test
Brown:						
Assets	$100,000 00					
Liabilities	80,000 00					
Net	$ 20,000 00	$20,000 00	$20,000 00	$20,000 00	$20,000.00	$20,000.00
Stock of Coulter (20% int.)	10,000 00	17,200 00	17,488 00	17,499 52	17,499 98	17,500 00
Total	$ 30,000 00	$37,200 00	$37,488.00	$37,499.52	$37,499 98	$37,500.00
Coulter:						
Assets	$100,000 00					
Liabilities	20,000 00					
Net	$ 80,000 00	$80,000 00	$80,000 00	$80,000 00	$80,000 00	$80,000 00
Stock of Brown (20% int.)	6,000 00	7,440 00	7,497 60	7,499 90	7,500 00	7,500 00
Total	$ 86,000 00	$87,440 00	$87,497 60	$87,499 90	$87,500 00	$87,500 00

Explanation. It will be noted that Brown owned $10,000 of Coulter's $50,000 capital stock, or 20%. The first test shows Brown to have a value of $30,000. Coulter's net worth, exclusive of the investment in Brown, is $80,000. If Brown is worth $30,000, then Coulter's investment in Brown is worth $6,000, and Coulter's net worth on the first test is $86,000. On the second test, Brown's investment in Coulter is 20% of Coulter's value as found in the first test, or $17,200. If Brown is worth $37,200, then Coulter's investment in Brown is 20% of $37,200, or $7,440, and Coulter's worth as shown by the second test is $87,440. This procedure is continued until the values of the investment of one in the other become equal, as in the fifth and sixth tests; that is, the value of Brown's investment in Coulter is approximately the same in the fifth and sixth tests, $17,500, and the value of Coulter's investment in Brown is the same in the fifth and sixth tests, $7,500.

Having found the net worth of Brown and of Coulter, proceed to find the net worth of Ames as follows:

Ames:	
Assets	$200,000 00
Liabilities	100,000 00
Net	$100,000 00
Stock of Brown—60% of $37,500	22,500 00
Stock of Coulter—60% of $87,500	52,500 00
Total	$175,000 00

Problems

1. A corporation wishes to create an insurance fund equal to 25% of the net profits after deduction of the insurance fund and of the manager's bonus of 10% of the profits, the bonus and the insurance fund both to be considered as expenses. The profits are $47,250. Find the amount of the bonus, the insurance fund, and the balance to be carried to surplus.

2. The assets and liabilities of two companies are as follows:

	Smith Company	*Jones Company*
Assets, exclusive of intercompany investments	$360,000	$320,000
Due from Jones Company	20,000	
Due from Smith Company		80,000
Deficit	100,000	160,000
	$480,000	$560,000
Liabilities, exclusive of intercompany investments	$400,000	$540,000
Due to Jones Company	80,000	
Due to Smith Company		20,000
	$480,000	$560,000

Both companies having failed, you are required to state how many cents on the dollar each firm can pay.

3. Three companies, M, N, and O, agree to consolidate. Their balance sheets show assets as follows:

$$\begin{aligned} M&: \text{Other assets, } \$\ 425{,}000 + 10\%N + 15\%O \\ N&: \text{Other assets, } \quad 462{,}500 + 15\%M + 15\%O \\ O&: \text{Other assets, } 1{,}145{,}000 + 12\tfrac{1}{2}\%M + 5\%N \end{aligned}$$

What are the assets of each company?

CHAPTER 26

Logarithms

Uses of logarithms. It is not the purpose of this chapter to explain how logarithms are derived, but to make as clear as possible the simple use of logarithms by the accountant. The accountant desires to know how to make a particular calculation accurately and in the least possible time. Logarithms are an exceedingly valuable tool for this purpose and have many practical applications.

The use of logarithms greatly simplifies the multiplying and dividing of numbers, the raising of numbers to powers, and the finding of the roots of numbers. Logarithms reduce the multiplying and dividing of numbers to problems of addition and subtraction; the finding of the power of a number to a problem in multiplication; and the extraction of a root to a problem in division.

Exponents. Logarithms are exponents; that is, the logarithm of a number is the exponent indicating the power to which a base must be raised to produce the number. The base of the common system of logarithms is 10. Therefore:

The logarithm of 100 is 2, because 10 must be raised to the 2nd power to produce 100.

The logarithm of 1,000 is 3, because 10 must be raised to the 3rd power to produce 1,000.

The logarithm of 10,000 is 4, because 10 must be raised to the 4th power to produce 10,000.

The logarithm of 100,000 is 5, because 10 must be raised to the 5th power to produce 100,000.

Obviously, the log of a number between 10 and 100 will be something between 1 and 2; for instance, the log of 50 is 1.69897. And obviously, the log of a number between 100 and 1000 will be something between 2 and 3; for instance, the log of 625 is 2.79588. A table may now be made as follows:

Number	*Log*
10	1.00000
50	1 69897
100	2 00000
625	2 79588
1000	3 00000
10000	4 00000
100000	5.00000

Parts of a logarithm. The part of a logarithm at the left of the decimal point is called the *characteristic;* the part at the right of the decimal point is called the *mantissa.*

If the number is 10 or more but less than 100, the characteristic is 1.
If the number is 100 or more but less than 1000, the characteristic is 2.
If the number is 1000 or more but less than 10000, the characteristic is 3.

Numbers which are less than 1 have negative characteristics. The nature of positive and negative characteristics is indicated below:

Number	*Characteristic of Log*
100,000	5
10,000	4
1,000	3
100	2
10	1
1.	0
1	−1
01	−2
.001	−3
.0001	−4

From the foregoing, the following rules for the determination of the characteristic may be developed:

Logs for the number 1 and all numbers in excess thereof have positive characteristics; the characteristic is one less than the number of figures at the left of the decimal point in the number.

Logs for numbers less than 1 have negative characteristics; the characteristic is one more than the number of zeros between the decimal point and the first significant figure at the right thereof.

The usual way of writing the logarithm of a number is as follows:

$$\log 118 = 2.071882$$

Sometimes a sign, "*nl*," is used to separate a number from its logarithm; the number is then read: 118 *nl* (the number whose logarithm is) 2.071882; or reversed, it is read: 2.071882 *ln* (the logarithm of the number) 118.

Characteristic. The characteristic, as has been stated, is the part of the number at the left of the decimal point of the logarithm. The characteristic of the logarithm of each number from 1 to 9 inclusive is 0.; from 10 to 99 inclusive, it is 1.; from 100 to 999 inclusive, 2.; and so forth. Also, the characteristic of each number from .1 to .9 is −1.; from .01 to .09 inclusive, −2.; and so forth.

Examples

The logarithm of	5.	has a characteristic of	0.
" " "	25.	" " " "	1.
" " "	490.	" " " "	2.

The logarithm of	370.	has a characteristic of	2.
" " "	4.	" " " "	0.
" " "	2.7	" " " "	0.
" " "	.3	" " " "	−1. or 9. (mantissa) −10
" " "	.49	" " " "	−1. or 9. " −10

Positive characteristic. From the foregoing list of numbers and of the characteristics of their logarithms, it will be seen that the characteristic for the log of 490. is the same as that for the log of 370. Also, it is the same for the log of 4. as for the log of 2.7. It is not the value of the digits in the number, but the number of digits at the left of the decimal point in the number, that gives the value to the positive characteristic.

Negative characteristic. If a number is less than 1., its log has a negative characteristic; that is, the log of .3 or of .49 has a negative characteristic of −1., written as 1., and the log of .01 or of .0245 has a negative characteristic of −2.

Examples

The log of .05	2 6990
The log of .0005	4 6990
The log of .000005	6 6990

If a given quantity is added to any number, and from the sum is subtracted the same quantity, the result is the same as the original number; that is, if 10 be added to 4, making 14, and from that sum 10 be subtracted, the result is still 4, although the form in which it is written is different; as, 4 = 4 + 10 − 10. Instead of writing a logarithm with a negative characteristic as, 1.1959 in some calculations it is found to be more convenient to indicate the negative characteristic in the following manner: 9.1959 − 10. Or it may be convenient in some cases to use a larger number; thus, $\bar{1}$.1959 may be represented as 29.1959 − 30, or as 6.1959 − 7. Any number may be added, provided the same number is used as a negative quantity. The change in form does not change the value.

A characteristic may be either positive or negative, but a mantissa is always positive. The small dash *over* the characteristic is intended to serve as a reminder that only the characteristic is a minus quantity, while the mantissa is invariably a plus quantity.

Mantissa. The mantissa of the log of a number is a group of figures which stands for or represents the sequence of the digits of the number.

The mantissa of the log of 125. is shown by the table to be .096910, which is also the mantissa of the logs of 12.5, 1.25, and 125,000. The mantissa is determined by the sequence of the digits of a number, and the characteristic is used to show the correct placing of the decimal point.

Examples of Numbers and the Mantissas of Their Logarithms

The sequence of digits in	125.	is indicated by the mantissa	.096910
" " " " "	1250.	" " " " "	.096910
" " " " "	125000.	" " " " "	.096910
" " " " "	12.5	" " " " "	.096910
" " " " "	1.25	" " " " "	.096910
" " " " "	.125	" " " " "	.096910
" " " " "	475.	" " " " "	.676694
" " " " "	625.	" " " " "	.795880
" " " " "	920.	" " " " "	.963788
" " " " "	1.	" " " " "	.000000

Examples of Numbers and Their Complete Logarithms

(*Characteristic and Mantissa*)

log	1235.	= 3.09167
"	123.5	= 2.09167
"	12.35	= 1.09167
"	1.235	= 0.09167
"	.1235	= 1.09167
"	.01235	= 2.09167
"	.001235	= 3.09167

The table given in Appendix III of this book is called a *six-place* table. This means that the mantissas as given are accurate to the sixth place. However, it does not necessarily follow that by the use of a six-place table calculations can be performed accurately to the sixth place. Tables of logarithms accurate to six, eight, or even ten places are sometimes used, but the six-place table is sufficient for ordinary purposes. If the accountant has many computations involving large numbers, he should procure a more extended table.

How to use a table of logarithms.

For numbers of one significant figure. The table shows only the mantissa of each logarithm. If it is desired to find the mantissa of a number such as 2, 20, 200, 2,000, or of any number whose only significant figure is 2, it is necessary to turn to the table and in the column at the left, headed "N," run down the line until the number 200 is reached. To the right of this number, in the column headed "0," the mantissa .301030 is found. This is the mantissa for the logs of 2, 20, 200, or .2, .02, .002, and so forth. The mantissa for

any other number of one significant figure may be found in a similar manner. In order to obtain the complete logarithm of a number it is necessary to supply the characteristic. The characteristic of the log of 2 is 0. Hence, the complete logarithm of 2 is 0.301030.

For numbers of two significant figures. If it is desired to find the logarithm of a number containing two significant figures as, 17, 170, or 1.7 it is necessary to look in the column at the left of the table, headed "*N*," and run down the column until 170 is reached. In the column to the right of 170, headed "0," the mantissa .230449 is found. Keep in mind that this is the mantissa for the logs of 17, 170, 1,700, and so forth.

For numbers of three significant figures. Assume that the logarithm of 118 is desired. In the left-hand column of the table, headed "*N*," find the number 118. To the right of this number, in the column headed "0," the mantissa .071882 is given. This, of course, is the mantissa for the logs of 118, 1.18, 11,800, and so forth. In the foregoing illustrations, the mantissa only was found. By the rules previously given, the characteristic of the log of 118 is ascertained to be 2. Therefore, the logarithm of the number 118 is 2.071882.

For numbers of four significant figures. To illustrate: It is required to find the logarithm of 1,648. Find the number 164 at the left of the table, in the column headed "*N*." On the horizontal line to the right of 164, in the column headed "8," the mantissa found is .216957. The characteristic for the log of 1,648 is 3. The complete logarithm of 1,648 is 3.216957.

Interpolation for numbers of five or more significant figures. The logarithm of a number of five or more significant figures may be found by the process of interpolation. This method is based upon the assumption that the differences of the mantissas are proportional to the differences of the numbers given. This proportion is not strictly exact, for the differences really grow smaller as the mantissas themselves grow larger. However, the results obtained deviate only slightly from the true results, and are sufficiently accurate for most purposes.

Example

Find the logarithm of 131,525.

Solution

In the column headed "*D*," on the line horizontal to the number 131 in the column headed "*N*," the difference between the mantissa of the log of 1,315 and the mantissa of the log of the next higher number, 1,316, is given as 330; this is more correctly stated as .000330. The excess of 131,525 over 131,500 is $\frac{25}{100}$ of the difference between 131,500 and 131,600. Therefore, multiply .000330 by

$\frac{25}{100}$ to obtain the fractional part of the difference in the mantissa; it is .0000825. The mantissa for 131,500 is .118926. To this add the .0000825, which will give the mantissa for 131,525, .1190085. Stated again:

The log of 131,600.	has a mantissa of	.119256
" " " 131,500.	" " "	" .118926
The difference of 100	= the difference of	.000330
131,525 − 131,500 = 25		

The difference of 25 in the numbers requires that a proportional part of the difference in the mantissas, .000330, be added to the mantissa of the log of the smaller number.

$$\tfrac{25}{100} \text{ of } .000330 = .0000825$$
$$.118926 + .0000825 = .1190085$$

.1190085 is approximately the correct mantissa. The characteristic for the log of 131,525 is 5. Therefore, the complete log of 131,525 is 5.1190085.

Problems

Find the logarithms of the following (express your answers in the form: "log 118 = 2.071882").

1. 4	**6.** 5	**11.** 1.127	**16.** .82378
2. 20	**7.** 82	**12.** 1,275	**17.** .03264
3. 30	**8.** 775	**13.** 1,482	**18.** .000382
4. .06	**9.** 827	**14.** 739.82	**19.** 1.00375
5. 25	**10.** 8.37	**15.** 68,439	**20.** 2 48765

To find a number when the logarithm is given. If the above process of finding the logarithm from a number is reversed, the number can easily be found from the logarithm. This process is called finding the antilogarithm. It is necessary first to find the digits of the number, and this must be done from the mantissa. If the mantissa can be found in the table, take the digits corresponding to the mantissa, and point off these digits decimally as indicated by the characteristic.

Example

Find the number whose logarithm is 0.281033.

Solution

In the table, the mantissa, .281033, is found in the vertical column headed "0," opposite the number 191. This indicates that the sequence of digits, ignoring possible initial or final zeros, is 191. Using the characteristic for the correct placing of the decimal, the number is found to be 1.91.

To find a number whose mantissa is not in the table. If the mantissa is not given in the table, it is necessary to reverse the process of interpolation.

Example

Find the number whose logarithm is 5.1190085.

Solution

In the table, the mantissa for the number nearest the one given, and less than it, is found to be .118926. Subtracting this from the mantissa given in the problem, the difference is .0000825. Use this as the numerator, and the difference, .000330 (indicated in the table in the column at the right, headed "*D*") as the denominator. Reduce this fraction .0000825/.0003300; the result is .25, or 25/100 of the difference between 1,315 and 1,316. Annex this amount to 1,315, and the sequence of digits is found to be 131,525, which, when pointed off as indicated by the characteristic, gives the result, 131,525.

Problems

Find the numbers represented by the following logarithms:

1. 1.658011	**6.** 1.698970	**11.** 8.172019 − 10	**16.** 4.637540
2. 2.711807	**7.** 2.681784	**12.** 19.003891 − 20	**17.** 2.682085
3. 0.681241	**8.** 3.624076	**13.** 3.176091 − 4	**18.** 6.463794 − 7
4. 9.672098 − 10	**9.** 5.638988	**14.** 6.432488 − 9	**19.** 3.390791
5. 2.707570	**10.** 1.800236	**15.** 7.363048	**20.** 1.844334

Rules for computation by logarithms.

RULE 1. To multiply numbers, add their logarithms; the sum is the logarithm of the product.

RULE 2. To divide numbers, subtract the logarithm of the divisor from the logarithm of the dividend; the remainder is the logarithm of the quotient.

RULE 3. To obtain a power of a number, multiply the logarithm of the number by the exponent of the power sought; the product is the logarithm of the power of the number.

RULE 4. To obtain a root of a number, divide the logarithm of the number by the index of the root sought; the quotient is the logarithm of the root of the number.

Multiplication by logarithms.

Example

Multiply 635 by 22, using the method of Rule 1.

Solution

log 635	2.802774
log 22	1.342423
log of product	4.145197

In the table, the mantissa .145196 is found to correspond to the digits 1,397. The characteristic 4 indicates that the product has five digits to the left of the decimal point, hence, the product is 13,970.

Problems

Multiply:

1. 25 by 25
2. 42 by 37
3. 240 by 381
4. 762 by 431
5. 42.5 by 49.2
6. 34.7 by 1.42
7. 1,430 by .249
8. 1.43 by .032
9. 1,480 by .138
10. 92.7 by 8.75
11. 3.39 by 8.92
12. 9.293 by 48.67
13. 143.9 by 1.478
14. 1.278 by 3.84
15. 145.3 by 6.296
16. .003 by .002
17. 100.05 by 100.25
18. 47.2 by 200
19. .999 by 647.2
20. 3.8 by 4.9

Division by logarithms. Rule 2 gives the procedure for division by means of logarithms.

Division of a greater number by a lesser number.

Example

Divide 875 by 37.

Solution

log 875 . . .	2.942008
log 37	1.568202
log quotient	1.373806

In the table, the mantissa, .373806, corresponds to the digits 23,648. The characteristic indicates that the quotient has two digits to the left of the decimal point. Pointed off, the answer is 23.648.

Division of a lesser number by a greater number. At times it is necessary to divide a smaller number by a larger one; this, of course, produces a quotient which is less than 1, and requires the operation of subtraction of a greater logarithm from a lesser one. To do this, change the form of the logarithm of the minuend; that is, add 10 (or a multiple of 10) to the characteristic of the minuend, and write -10 (or a multiple of -10, the same multiple to be used in each case) after the mantissa.

Example

Divide 269 by 239,000.

Solution

log 269 = 2.429752, or	12.429752 − 10
Deduct log 239,000, or . . .	5.378398
log quotient . . .	7.051354 − 10

Changing 7.051354 − 10 to its simple form, it becomes $\bar{3}.051354$. By reference to the logarithm table of mantissas, and by applying the rule for pointing off numbers by the characteristic, the quotient is found to be .0011255 (correct to five significant figures).

The above method is also applicable to the subtraction of a negative logarithm from a positive one.

Example

Divide 14,200 by .000191.

Solution

log 14.200 = 4.152288, or....................	14.152288 − 10
log .000191 = $\bar{4}$.281033, or............ ...	6 281033 − 10
log quotient......................	7.871255 − 0

By finding the antilog of 7.871255, the resulting quotient is ascertained to be 74,345,500 (correct to five significant places).

Problems

Divide:

1. 128 by 64	**8.** 2.486 by 3.45	**15.** 1.425 by 892.7
2. 2,160 by 150	**9.** .6843 by 89	**16.** 147.25 by 9,276
3. 344 by 8	**10.** 9.278 by 12.43	**17.** .03 by 6,000
4. 93 by .31	**11.** 6 by 2	**18.** .0125 by 3,427
5. 649.4 by 24.3	**12.** 89 by 47	**19.** 1.005 by 927.8
6. 8.42 by 2.48	**13.** 2.1 by 48	**20.** 2.4255 by 384.275
7. .3472 by 124	**14.** 3.875 by 238.7	**21.** 6,497.8 by 2.874

Powers of numbers. Rule 3 gives the procedure for finding the powers of numbers.

Example

Find the fourth power of 26.

Solution

log 26	1 414973
Multiply by the exponent of the power	4
log power .	5 659892

In the table, the mantissa .659892 represents the sequence of digits 456,975. The characteristic 5 indicates that the product has six digits to the left of the decimal point. Hence, the fourth power of 26 is 456,975 (correct to five significant places).

Process with a negative characteristic.

Example

Find the fifth power of .025.

Solution

log .025 = $\bar{2}$.397940, or	8 397940 − 10
Multiply by the exponent of the power.. .	5
	41 989700 − 50
41.989700 − 50, changed .	$\bar{9}$ 9897
The mantissa .9897 .	976562
Pointed off by the characteristic . .	00000000976562

Problems

Find the value of:

1. 5th power of 25	**8.** 12th power of 7	**15.** 8.92^4
2. 4th power of 35	**9.** 12^3	**16.** 146^5
3. 5th power of 2	**10.** 14^2	**17.** $.07^2$
4. 7th power of 1.25	**11.** $9{,}200^6$	**18.** $.97^5$
5. 4th power of 2.47	**12.** 18^3	**19.** 30^3
6. 3rd power of 575	**13.** 147^2	**20.** 27^4
7. 9th power of 4	**14.** $.015^2$	**21.** 9^5

Roots of numbers. Rule 4 gives the procedure for finding the roots of numbers.

Example

Find the cube root of 875.

Solution

log 875	2.942008
Divide by index of the root	3)2.942008
log quotient	0.980669

In the table, the mantissa .980669 corresponds to the sequence of digits 95,646, and the characteristic indicates that the number is 9.5646.

Process with negative characteristics. In stating the equivalent of the negative characteristic of a logarithm, care must be taken to see that the right-hand number, or minus quantity, is exactly divisible by the index of the root with a quotient of 10 or a multiple thereof.

Example

Find the 4th root of .125.

The negative characteristic of the complete logarithm of .125 may be stated in several different ways: as $\bar{1}$.096910; as 3.096910 − 4, as 7.096910 − 8, as 19.096910 − 20; or as 39.096910 − 40.

In this problem, in order to avoid any complication, it is best to have the right-hand number of the characteristic exactly divisible by 4 (the index of the required root), with a quotient of 10.

Solution

log .125	$\bar{1}$.096910
Or, log .125	39.096910 − 40
Divide by index of root	4)39.096910 − 40
log quotient	9.774228 − 10
Or	$\bar{1}$.774228

The mantissa .774228 corresponds to the succession of digits 594,604 (accurate to five places). Pointed off by the characteristic, −1, the 4th root of .125 is .594604.

Problems

Find the value of the following to five significant places:

1. Square root of 64.
2. Square root of 97.
3. Square root of .64.
4. Cube root of 81.
5. Cube root of .081.
6. 6th root of 49.
7. 7th root of 750.
8. 10th root of 10.
9. $\sqrt[3]{125}$.
10. $\sqrt[10]{1}$.
11. $\sqrt[3]{3.89}$.
12. $\sqrt[5]{89.27}$.
13. $\sqrt{.9643}$.
14. $\sqrt[10]{2980}$.
15. $\sqrt[3]{42^2}$.
16. $\sqrt[3]{19^2}$.
17. $\sqrt{9 \times 97}$.
18. $\sqrt[3]{\frac{47}{929}}$.
19. $506 \times \sqrt[4]{40}$.
20. $3.87^2 \times \sqrt[3]{\frac{1}{2}}$.
21. $\sqrt[5]{225} \times \sqrt[10]{781}$.

The slide rule. This is an old device, yet it is used to only a limited extent by accountants in general. It has been described as "logarithms on a stick." In its simple form, this mechanical

device consists of a grooved base rule into which a slide rule is fitted. A third part is the runner. The graduations on the upper part of the base and slide are called "upper scale," while those on the lower part of the base and slide are called "lower scale." These graduations are for different purposes, and because of the different requirements they are graduated differently.

Use of slide rule. The slide rule is used either to check figures or for original calculations. It may be used to check or compute any operation involving multiplication, division, raising to powers, or extracting of roots. It has a great many applications in business, although it is generally considered as a device used only by engineers.

The slide rule should appeal to the accountant as a device which may be carried in the pocket or the brief case; its weight is negligible.

Accuracy of calculations made by the slide rule. It is possible, after having attained proficiency in the handling of the slide rule, to obtain results in which the margin of error will not be more than one-quarter of 1%. This is satisfactory for most business problems. As in logarithms, when the slide rule is used the digits are taken from the left to the right of a number, regardless of the value of the number. The slide rule is as nearly accurate in the calculation of decimal numbers as it is in the calculation of whole numbers of large denominations. It will give correct answers of two places, and if careful computations have been made, a three-place solution of a good degree of accuracy may be expected. If extreme care is used in the computation of a problem, an answer of four places may be had with a fair degree of accuracy. Of course, a long and carefully graded slide rule will give better results than either a short or a carelessly graded rule.

Theory of the slide rule. The theory of the slide rule is indicated very roughly in the following simple example and illustration.

Example

Find the sum of 4 and 6.

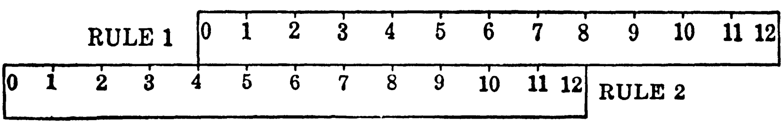


Figure 2.

Solution

Above are two ordinary rulers, set opposite each other. To find the sum of 4 and 6, perform the following steps:

(1) Set the "0" on Rule 1 over the "4" on Rule 2.
(2) Observe the figure "6" on Rule 1.
(3) On Rule 2, immediately below the "6" on Rule 1, will be found "10"—the sum of 4 and 6.

The process of subtraction may be shown as follows.

Example

Find the difference between 9 and 4.

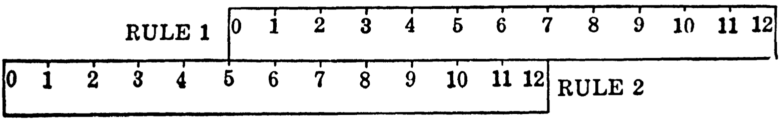


Figure 3.

Solution

(1) Locate the subtrahend "9" on Rule 2.
(2) On Rule 1, locate the minuend "4," and place it immediately over the subtrahend "9" on Rule 2.
(3) On Rule 2, the number immediately below the "0" on Rule 1 will be the remainder, which in this case is 5.

The foregoing examples show that addition and subtraction of small numbers can be performed on two ordinary rulers. The principle of the slide rule is similar.

On the common slide rule, the graduations are made according to logarithms. Hence, if any two numbers on the slide rule are added, the result obtained is the sum of two logarithms.

The addition of the logarithms of numbers results in multiplication of the numbers; and the subtraction of the logarithms of numbers results in division of the numbers.

How to learn to use the slide rule. Practice is without doubt the only efficient method of learning how to use the slide rule. If possible, a slide rule should be obtained for use in this chapter. If, however, a slide rule is not available, a cardboard model may be made for practice. Care must be used to have the markings as accurate as possible, in order to obtain fair results. A model such as the following can be used very conveniently for the practice material.

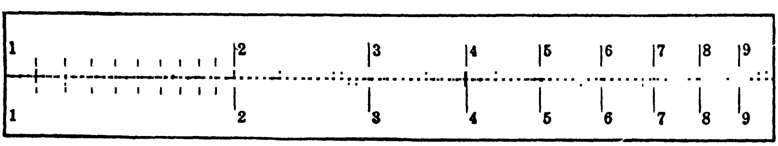


Figure 4.

Reading the slide rule. In reading the numbers, go over the rule from left to right, as follows: 1, 2, 3 . . . 10; then, beginning at 1 again and calling it 10, read 10, 20, 30 . . . 100; then, again beginning at 1 and calling it 100, read 100, 200, 300 . . . 1,000. It is possible to do this because the mantissa for 10 is the same as that for 100, 1,000, etc.

It will be noticed in Figure 4 that the spaces decrease from left to right. These decreases should correspond exactly to the differences between the logarithms from 1 to 10. Assume that you divide your model rule into 1,000 equal parts. Then, since log 2 = .301, the 2 would be placed at the 301st graduation. Log 3 = .477; therefore 3 would be placed at the 477th graduation. Log 4 = .602; therefore 4 would be placed at the 602nd graduation. Similarly, 5 would be placed at the 698th; 6 at the 778th; 7 at the 845th; 8 at the 903rd; and 9 at the 954th.

Marks can be put in to show the mantissas for the logarithms of 1.5, 2.5, 3.5, and so on, but they will not be half the distance between the previous graduations, because log 1.5 is 0.176, and this is not half the difference between log 1 and log 2.

It will be noticed that the distance from 1 to 2 is divided into 10 divisions. These are read from left to right, like telephone numbers, thus: one-one, one-two, one-three, and so forth, to one-nine; then 2. These are understood as 1.1, 1.2, 1.3, and so forth. Consulting the table of logarithms for the logs of 110, 120, 130, and so forth, we find that the marks will be placed at the following graduations: 41, 79, 113, 146, 176, 204, 230, 255, and 278.

Construction of model slide rule. Obtain a piece of cardboard $12\frac{1}{2}$ inches long and 1 inch in width. Rule a line midway along the full length, and mark it off in graduations of one-eighth inch. This will make a measure with 100 graduations instead of 1,000, but for the purpose of this work it will be satisfactory.

Since log 2 = .301, mark 2 at 30.1
log 3 = .477, mark 3 at 47.7
log 4 = .602, mark 4 at 60.2
log 5 = .698, mark 5 at 69.8
log 6 = .778, mark 6 at 77.8
log 7 = .845, mark 7 at 84.5
log 8 = .903, mark 8 at 90.3
log 9 = .954, mark 9 at 95.4

Between 1 and 2 are 1.1, 1.2, 1.3, . . . 1.9, as previously explained.

Since log 1.10 = .041, mark at 4.1
log 1.20 = .079, mark at 7.9
log 1.30 = .113, mark at 11.3
log 1.40 = .146, mark at 14.6
log 1.50 = .176, mark at 17.6
log 1.60 = .204, mark at 20.4
log 1.70 = .230, mark at 23.0
log 1.80 = .255, mark at 25.5
log 1.90 = .278, mark at 27.8

For closer graduations, you will find that you can make 10 indentations with a sharp pin in one-eighth inch space. By carefully counting the points placed, you can use the full logarithm and make a fairly accurate slide rule.

Having completed the graduations, cut the cardboard lengthwise on the medial line. This will give two pieces with measurements exactly the same. These two measures may now be used as Rule 1 and Rule 2 in the following simple problems.

Multiplication on the slide rule. Add the logarithms of the numbers to be multiplied.

Example

Multiply 2 by 3.

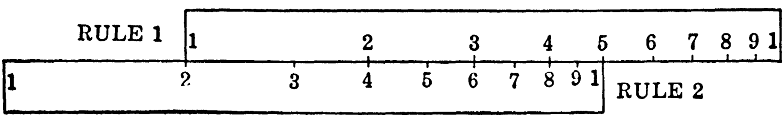


Figure 5.

Solution

(1) Locate "2" on Rule 2.
(2) Place "1" on Rule 1 over "2" on Rule 2.
(3) Locate "3" on Rule 1.
(4) Read the number on Rule 2 immediately below, which is "6."

Problems

Multiply:

1. 2 by 4. **2.** 3 by 4. **3.** 2 by 5. **4.** 3 by 2. **5.** 4 by 2. **6.** 3 by 3.

If the problem is of such a nature that the rules cannot be operated by placing the left-hand "1" on Rule 1 over the number on Rule 2, it is necessary to use the right-hand "1" on Rule 1.

Example

Multiply 4 by 5.

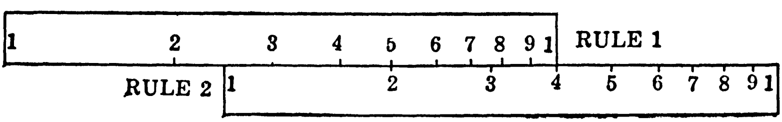


Figure 6.

Solution

(1) Locate "4" on Rule 2.

(2) Place "1" on Rule 1 over "4" on Rule 2, so that the "5" on Rule 1 is over Rule 2. In this case it is necessary to use the "1" on the right-hand side of Rule 1.

(3) Read the number immediately under "5" on Rule 1, which is "2." "2" may be either 2., .2, .02, 20, 200, or any other number in which the left-hand digit is "2." In this case the number can be determined by inspection to be 20.

Problems

Multiply:

1. 5 by 8. **2.** 4 by 5. **3.** 3 by 10. **4.** 20 by 20. **5.** 40 by 5. **6.** 2 by 30.

Division on the slide rule. Subtraction of logarithms results in the division of the numbers.

Example

Divide 9 by 6.

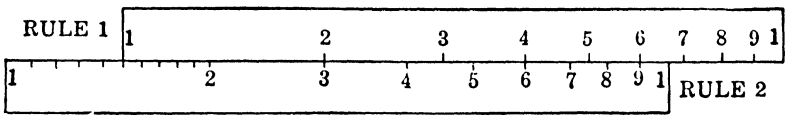


Figure 7.

Solution

(1) Locate "9" on Rule 2.

(2) Place "6" on Rule 1 over "9" on Rule 2.

(3) Read on Rule 2 immediately below "1" on Rule 1. As the "1" is over the 5th graduation on Rule 2, the digits will be 15, and by inspection the answer is determined to be 1.5.

Problems

Divide:

1. 6 by 2.	**4.** 6 by 4.	**7.** 35 by 7.	**10.** 95 by 5.
2. 8 by 4.	**5.** 9 by 4.	**8.** 25 by 5.	**11.** 18 by 12.
3. 5 by 2.	**6.** 64 by 8.	**9.** 12 by 4.	**12.** 36 by 18.

Another type of problem is that of multiplying two or more numbers and dividing their product by another number.

Example

$8 \times 9 \div 4 = ?$

Solution

(1) Set Rule 1 so that the "1" on the right is over "8" on Rule 2.

(2) Read the number immediately under "9" on Rule 1, which is "72."

(3) Set "4" on Rule 1 over "72" on Rule 2.

(4) Read the number on Rule 2 immediately under "1" on Rule 1, and the answer is found to be 18.

Problems

Solve the following by the use of the slide rule:

1. $16 \times 6 \div 32$.	**3.** $35 \times 35 \div 5$.	**5.** $37 \times 19 \div 13$.	**7.** $8 \times 14 \div 12$.
2. $20 \times 40 \div 8$.	**4.** $45 \times 12 \div 8$.	**6.** $44 \times 34 \div 27$.	**8.** $4 \times 7 \div 200$.

The following problems illustrate some of the practical applications of the slide rule.

Problems

1. *Payroll calculation.* Brown's time card for a particular day showed the following:

Job No.	Time H	Time M
12	3	30
21	2	10
32		50
45	1	30
	8	00

If Brown was paid 54¢ an hour, what was the labor cost chargeable to each job?

2. *Prorating expense.* The power cost of a small plant is to be distributed to the departments on the basis of horsepower hours, as follows:

Dept. A	45 horsepower
Dept. B	35 horsepower
Dept. C	90 horsepower
Dept. D	20 horsepower
Dept. E	5 horsepower
Dept. F	5 horsepower
Total	200 horsepower

The total power cost was $450. What was the cost of power in each department?

3. The air fare from X to Y is $40. The traveler goes over three divisions of airway, respectively 380, 230, and 190 miles in length. Find the amount of the fare to be apportioned to each division.

4. An article that cost $25 is sold for $50 less 20%. Find the per cent of gain on the cost. Find the per cent of gain on the selling price.

5. Given:

Sales	$500
Cost of Goods Sold	300
Selling Expenses	75
General Expenses	50
Profit	

What per cent of the sales is each item?

6. Work the following:

	Cost	% on Selling Price	Selling Price
(*a*)	$ 5 00	20	....
(*b*)	8 00	40	....
(*c*)	12 00	25	....
(*d*)	20 00	20	....
(*e*)	16 00	30	....

7. The list price of an article is $25, less 10% and 5%. Find the net cost

8. Find the interest on $600 at 5% for 3 years 6 months.

9. A field is 40 rods wide and 80 rods long. How many acres does it contain?

10. Find the cost of 80 items at $1.50 a dozen.

CHAPTER 27

Graphs and Index Numbers

Charts and graphs. Charts and graphs are becoming increasingly popular as a means of presenting the results of accounting and mathematical computations. Accountants, credit men, production managers, sales managers, advertising men, and general business executives are realizing more and more how greatly graphic charts may help them in their work. The reason is obvious. Long rows of figures must be thoroughly studied if the relations between quantities are to be grasped. This is a tedious task. On the other hand, pages of valuable data may be presented on a simple chart that will convey more real information than the most elaborately written report. It is necessary, however, to distinguish between important and unimportant data. Furthermore, a method of presentation must be chosen that will convey a correct impression, for it is quite possible to prepare misleading charts from correct data. Two important points must, therefore, be borne in mind:

(1) The selection of the data;
(2) The selection of the design.

Circle chart. This type of chart is used extensively for popular presentation, and is designed to exhibit the true proportions of the component parts of a group total. It is adapted to such purposes as exhibiting the distribution of disbursements, the sources of receipts, and the allocation of appropriations in government finance.

The circle with sectors, however, is not so desirable a form of presentation as the bar chart, described in later paragraphs, since it does not possess the same degree of flexibility. It is impossible, for instance, in a profit and loss analysis, to exhibit a loss. Moreover, it does not always permit a convenient arrangement of captions, which must sometimes be written in at an angle. Another disadvantage is that the figures are not easily compared. For these reasons, it is probably best to limit the circle chart to the illustration of facts which are not intended to be compared from

period to period. However, the sector method is so widely used that it is perhaps better understood generally than any other.

The circle is segmented on the basis of 100°, not 360°, as geographic circles are segmented. This is because the chart circle is designed to exhibit a percentage scale.

Example

Distribution of the expense dollar.

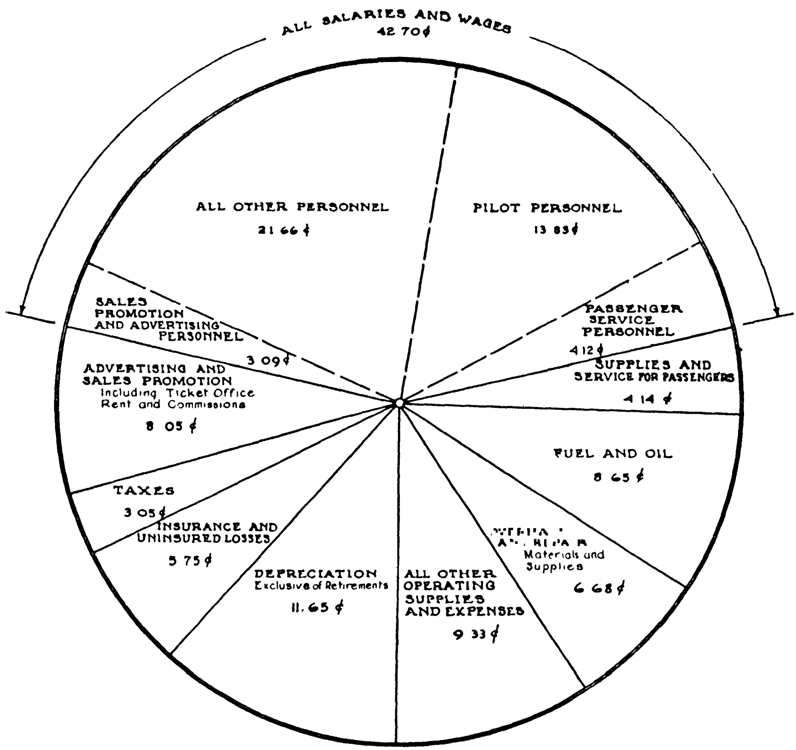


Figure 8. Circle Chart.

Comparison of circles. Comparisons in magnitude are sometimes made by presenting circles of different sizes. The objection to this method is the resulting confusion in the mind of the reader as to whether area or diameter is used as the basis of measurement.

It is impossible to estimate accurately the difference between the diameters of two circles by merely looking at them. When comparative diameters are being estimated, the circles themselves have to be subordinated in the mind of the reader while the diameters are visualized. Charts were devised because of their ease of comprehension, and their purpose is defeated when conflicting

metal processes are involved. For this reason, circles of different sizes should never be used for comparative purposes.

The same criticism applies to squares and cubes. A cube whose edge is twice that of another will possess eight times the cubic content and four times the outside area of the smaller one, and unless the basis of measurement is carefully explained, the comparison may easily be misleading.

Problems

1. Using the figures in the following condensed operating statement, prepare a circle chart showing the distribution of the "sales dollar."

Sales	$66,734.49
Plant Operating Expenses	21,464.91
Payroll	18,055 22
Taxes	1,055.07
Depreciation	14,252.63
Depletion	4,667.20
Net Profit	7,239 46

2. Prepare a circle chart to illustrate the following accounts receivable analysis.

Current to 60 days old	47 08%
90 days old	11.69%
120 days old	6 62%
4 to 6 months old	10.65%
6 months to 1 year old	8 54%
Over 1 year old	15.42%

3. Prepare a circle chart to illustrate the distribution of the sales dollar.

Raw materials	$ 55
Wages and salaries	.25
Direct taxes	.07
Selling, advertising, and miscellaneous expense	.05
Reinvestment in the business	.03
Wear and tear on equipment	.02½
Dividends	.02½

Bar chart. The bar chart, like the circle chart, is designed to exhibit the true proportions of the component parts of a group total.

Bars used in charting may consist of single heavy lines, or they may be widened into rectangles. Three ways of presenting data by means of bars are in common use:

(1) Comparisons are made by presenting a series of bars of different lengths, each bar representing a different magnitude (see Figure 9).

(2) A single bar is subdivided into component parts (see Figure 10).

(3) A combination of (1) and (2) may be employed (see Figure 11).

Where color is not used, bars or parts of a bar may be differentiated by crosshatching, and also by the use of solid black and white.

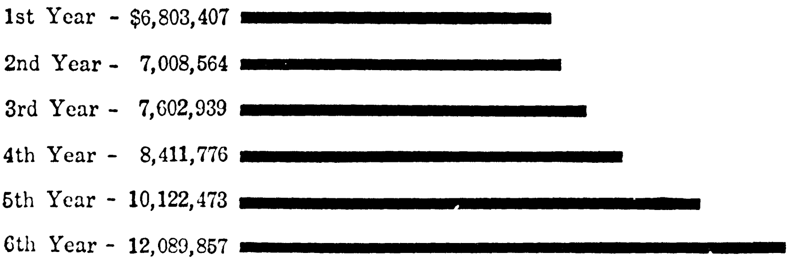


Figure 9. Bar Chart Showing Sales for 6-year Period.

Crosshatching consists of the fine parallel lines drawn across the face of the bar at various angles, and sometimes crossed into small rectangles.

In making comparisons by means of parallel bars, it is essential that the bars be of the same width, in order that measurement by length be not confused with that of area.

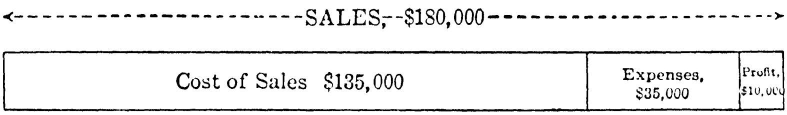


Figure 10. Bar Chart Showing Net Profit on Sales.

Bars may be placed in either horizontal or vertical position.

In the bar method of charting, figures may be placed at the side or in the bar and decimal points kept in line; it is thus easy to foot the figures representing the various components and to verify the total (Figure 11).

Because of its decimal divisions, a millimeter scale is most convenient in constructing these charts.

Problems

1. Chart the following data, using vertical bars.

Period	*Amount*
1	40,000
2	50,000
3	60,000
4	75,000
5	80,000
6	90,000
7	95,000
8	100,000
9	105,000
10	110,000

2. Chart the following data, using the single bar subdivided into component parts.

Sales	$1,000,000
Cost of Sales .	550,000
General Expenses . . .	200,000
Selling Expenses . .	150,000
Net Profit . .	100,000

3. Chart the following, using horizontal bars.

Period	*Amount*
1	2,931
2	9,052
3	13,541
4	16,403
5	20,919
6	28,063
7	36,365
8	41,393
9	49,404
10	56,615

4. Chart the following similarly to Figure 12.

GROWTH IN NUMBER OF EMPLOYEES

Year	*No. Employees*	*No. Male*	*No. Female*
1st	6,587	1,859	4,728
2nd	6,893	2,081	4,812
3rd	7,205	2,270	4,935
4th	7,581	2,317	5,264
5th	7,810	2,168	5,642
6th	8,104	2,278	5,826

Line or curve chart. The line or curve chart is probably adaptable to a greater variety of uses than any other type of graphic presentation. It is used particularly to exhibit trends and fluctuations in data, the abnormal conditions being shown by unusual "peaks" and "valleys."

On line charts, the scale is indicated by vertical and/or horizontal rulings. If the coördinate type of ruling is used, the lines in each direction are spaced an equal distance apart, horizontal lines marking the vertical scale, and vertical lines marking the horizontal scale.

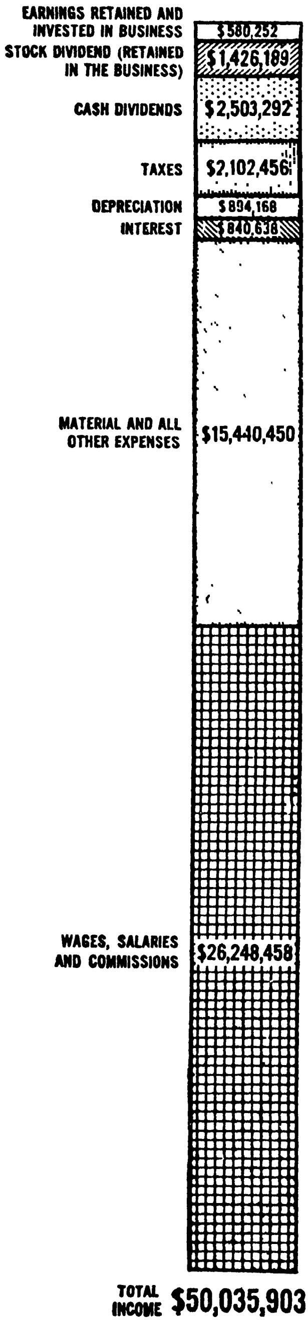


Figure 11.

All rectilinear charts have two axes—the horizontal, called the x-axis, and the vertical, called the y-axis.

Rules for coördinate charts. Custom has established certain rules governing the construction of coördinate charts, which must be observed if they are to be plotted in accordance with good usage.

(1) The zero line should always appear, or attention should be specifically called to its omission.

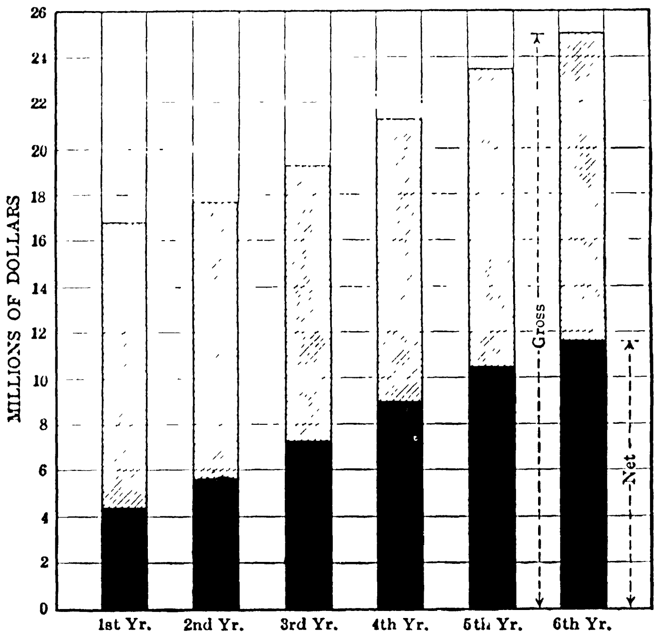


Figure 12. Bar Chart Showing Annual Gross and Net Income.

(2) The time element should always be expressed by the horizontal scale, and magnitude by the vertical scale.

(3) The curves should be sharply distinguished from the ruling.

(4) Figures and lettering should be so placed that they are read from the bottom or from the right-hand side.

(5) Exact data should be inserted at the top of the chart, the figures in each case appearing immediately above the corresponding point on the curve.

(6) The figures of the vertical scale should be placed on the left. In wide charts they may be repeated on the right.

(7) The horizontal scale should read from left to right, and the vertical scale from bottom to top.

It is considered good practice to make the zero line heavier than the other coördinate rulings. In percentage charts the 100% line is also accentuated by heavier ruling.

Example

The earnings of a corporation over a period of years were as follows:

1st year	$39,202,000
2nd year	37,555,000
3rd year	28,621,000
4th year	30,438,000
5th year	28,693,000
6th year	27,319,000
7th year	28,358,000
8th year	35,941,000
9th year	31,772,000
10th year	34,249,000

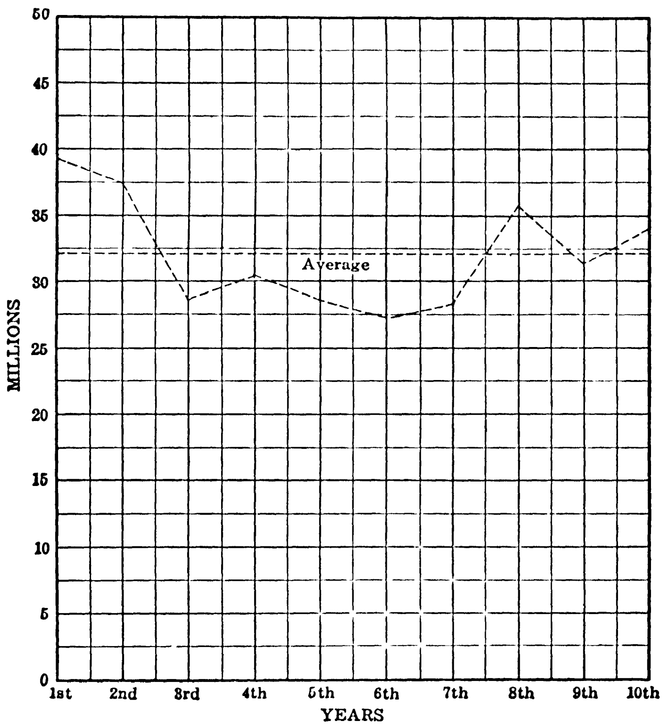


Figure 13.

NOTE: Limited space does not permit insertion of exact figures (Rule 5).

Problems

1. Using the following data, prepare a line chart showing the corporation's earnings and the dividends paid over a period of years.

Year	*Net Income*	*Dividends Paid*
1st	$28,154,431	$16,354,000
2nd	35,422,514	16,360,632
3rd	49,129,417	16,369,400
4th	28,684,916	16,404,509
5th	31,548,606	17,478,459
6th	32,070,274	18,209,281
7th	30,618,778	20,639,196
8th	32,600,150	20,662,854
9th	44,552,482	20,662,854
10th	35,419,903	20,943,094
11th	35,657,410	22,609,650

2.* From the information given in the following table, prepare a line graph of the Bonded Debt Limit and the Net Bonded Indebtedness of the City of X for the 15-year period. (Scale, 1 in. = $2,000,000.)

Year	*Assessed Valuation*	*Bonded Debt Limit*	*Net Bonded Indebtedness*
1st	$442,932,255	$15,887,062	$10,698,500
2nd	460,548,763	18,272,142	9,321,050
3rd	486,424,005	20,804,104	10,577,500
4th	496,342,170	23,291,794	11,101,500
5th	505,713,510	23,919,607	11,921,000
6th	521,239,125	24,702,675	14,730,750
7th	539,457,120	25,491,759	16,566,000
8th	574,020,559	26,367,724	16,534,750
9th	588,556,266	27,289,865	18,254,800
10th	675,611,540	28,988,846	22,030,250
11th	681,198,160	30,588,436	23,965,500
12th	677,070,755	27,750,500	24,800,000
13th	725,603,037	29,033,300	27,403,300
14th	755,229,851	30,773,800	25,023,500
15th	810,509,504	33,974,550	25,744,500

Logarithmic chart. The common logarithms of the Briggs' table, being the expressions of numbers in terms of the power of 10, are particularly adapted to the presentation of percentage relationships. This fact has led to the construction of a chart in which percentage relationships are revealed by a comparison of different sets of data plotted in terms of numerical magnitude.

The logarithmic chart is a variation of the rectilinear type. The ruling differs from that of the customary coördinate chart in that the data lines representing the scale are not evenly spaced, but conform to certain proportions expressed by the first ten numbers in a table of common logarithms.

Figure 14 illustrates the method of laying out such a scale. The first column of figures is purely for drafting purposes, and

* C. P. A., Wisconsin.

consists of the first ten figures of a logarithmic table, only the first two decimal digits being used in each case.

The condensed table of logarithms of numbers, page 274, will be of assistance in preparing a logarithmic chart.

In the vertical scale, each horizontal line is spaced the distance from the base of the scale which represents the proportion that its scale number bears to 100; that is, the first line above the base line is thirty one-hundredths of the total height of the scale,

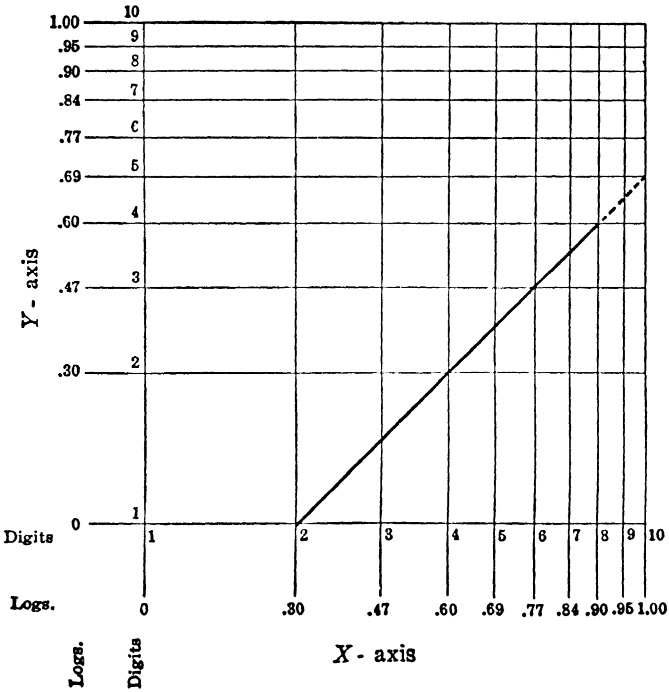


Figure 14. Logarithmic Chart.

the next line is forty-seven one-hundredths, and so on, the top line representing 100. A ruler having a 100-millimeter scale may be conveniently used for making these horizontal lines.

The second vertical column represents the numerical magnitude scale, and is numbered from 1 to 10, or some multiple thereof, the ruled lines being spaced according to the logarithms of these numbers. The result is that when data are plotted on such a chart in terms of numerical magnitude, the relationships shown between the various groups of data plotted will be correct percentage relationships. This does not hold true where data are plotted numerically on an ordinary coördinate chart; where the curves

represent widely differing magnitudes, an attempted comparison of the fluctuations in the data will be misleading. In order that data plotted on a coördinate chart may present correct percentage relationships, the lines must represent a percentage, not a numerical, scale.

LOGARITHMS OF NUMBERS

Number	*Logarithm*	*Number*	*Logarithm*	*Number*	*Logarithm*	*Number*	*Logarithm*
1	0.00	**10**	1 00	**100**	2 00	**1,000**	3 00
2	0.30	**20**	1.30	**200**	2.30	**2,000**	3 30
3	0.47	**30**	1.47	**300**	2 47	**3,000**	3 47
4	0.60	**40**	1.60	**400**	2 60	**4,000**	3 60
5	0.69	**50**	1.69	**500**	2 69	**5,000**	3 69
6	0.77	**60**	1.77	**600**	2 77	**6,000**	3 77
7	0.84	**70**	1 84	**700**	2.84	**7,000**	3 84
8	0 90	**80**	1 90	**800**	2 90	**8,000**	3 90
9	0.95	**90**	1 95	**900**	2 95	**9,000**	3 95
						10,000	4.00

In comparing on a logarithmic scale data of widely differing magnitudes, it is necessary to use more than one grouping of logarithmic rulings. Each such grouping is called a cycle, because it represents 10, or some multiple thereof. For instance, if data represented by figures in the hundreds group and data running into the thousands were to be compared, it would be necessary to use two cycles; if the figures ran into the ten-thousands group, it would be necessary to use three cycles.

It will be noticed that the base line in a logarithmic chart is numbered 1, instead of 0. This is because in the tables the log of 1 is .0. Therefore, in a logarithmic chart there is no zero line.

To illustrate the use of a full logarithmic chart, a simple example in multiplication may be cited. Applying the principle that in multiplication the logarithm of the product of two numbers is the sum of the logarithms of the numbers, the addition may be performed graphically on a logarithmic chart. By doubling the distance of any number represented by a digit, the square of the digit is obtained. Thus, the distance from 1 to 9 is twice the distance from 1 to 3, because 9 is the square of 3.

Since the scales of both axes are the same, lines projected at right angles from identically numbered points on both axes will complete a square, the diagonal of which is 45°, to the right of any point on the base of the x-axis; for instance, any horizontal line which intersects this diagonal will, if projected downward from the intersecting point to the base line of the x-axis, and at right

angles thereto, record on the x-axis digit scale the product of the two numbers representing the starting points of the diagonal and the horizontal lines.

Figure 14 illustrates a computation of this kind, and affords a mechanical demonstration of the logarithmic principle referred to. A diagonal line is projected upward at an angle of 45° from the

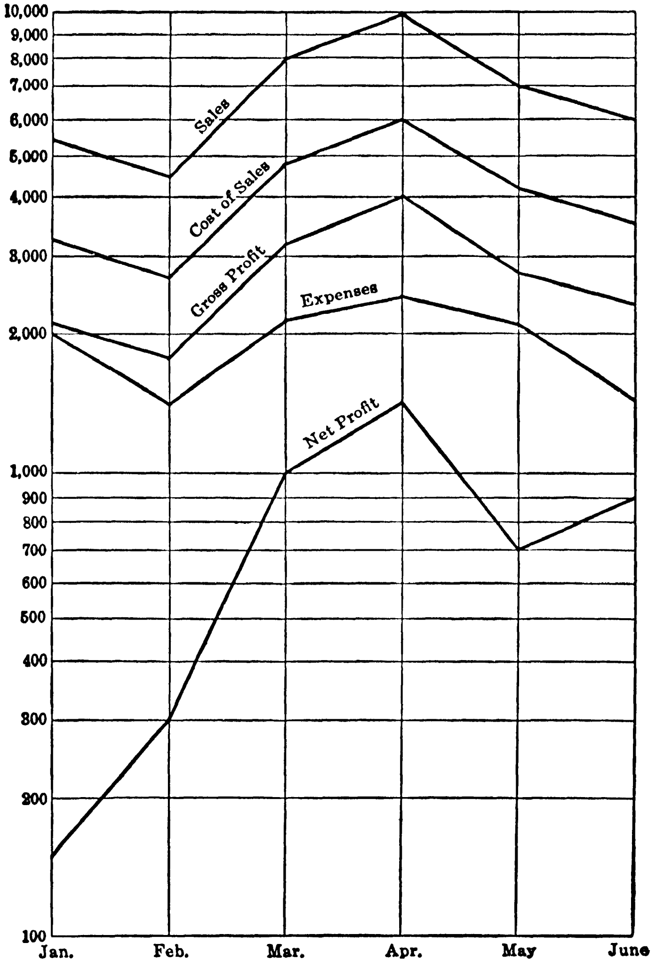


Figure 15. Ratio Chart.

digit 2 on the x-axis. The point at which it intersects the horizontal line numbered 4 on the y-scale is directly above digit 8 on the x-scale, and 2 times 4 is 8. Likewise, the sum of the distances 1 to 2 on the x-axis, and 1 to 4 on the y-axis, will, if laid off by a pair of dividers, arrive at point 8 on either scale.

Ratio charts. Figure 15 illustrates a two-cycle chart. Only the horizontal lines are ruled in accordance with the logarithmic scale. The vertical lines are evenly spaced. This is called a semi-logarithmic or ratio-ruled chart, and is the kind most commonly used.

The use of the full logarithmic chart, with the ratio ruling in both directions, is rare, and is confined chiefly to mathematical demonstrations.

The ratio chart has its limitations. While it exhibits correct percentage relationships, it does not record correct numerical magnitudes, and it should be used only where percentage relationships are desired. But because relationships between fluctuating data are more easily grasped in terms of percentages than in purely numerical terms, the semi-logarithmic chart has a wide field of usefulness.

Where the percentage of increase or decrease of one item, such as sales, is to be compared with the percentage of increase or decrease of some other item, such as expenses, the result is best shown in the ratio or semi-logarithmic chart.

Example*

	Sales	Cost	Gross Profit (40%)	Expense	Net Profit
January	$ 5,400	$ 3,240	$ 2,160	$ 2,000	$ 160
February	4,500	2,700	1,800	1,500	300
March ...	8,000	4,800	3,200	2,200	1,000
April	10,000	6,000	4,000	2,500	1,500
May..........	7,000	4,200	2,800	2,100	700
June..........	6,000	3,600	2,400	1,500	900
	$40,000	$24,540	$16,360	$11,800	$4,560

Problems

1. Prepare a ratio chart to illustrate the following condensed operating statements:

	1st Year	2nd Year	3rd Year	4th Year	5th Year	6th Year
Sales..........	$61,960.29	$74,401.38	$80,598.00	$72,887.60	$79,647.14	$65,315.48
Cost of Sales...	27,745.59	32,967.89	48,935.10	41,660.83	48,945.54	33,414.53
Gross Profit...	34,214.70	41,433.49	31,662.90	31,226.77	30,701.60	31,900.95
Expenses......	31,152.64	33,308.86	28,640.73	30,580.32	30,342.26	29,736.02
Net Profit ...	3,062.06	8,124.63	3,022.17	646.45	359.34	2,164.93

* Charted in Figure 15.

2. Prepare a ratio chart to illustrate the following comparative income accounts of a public utility company:

Year	Gross Earnings	Maintenance and Renewals	Power-Opr., Conducting Transportation, and General	Taxes	Fixed Charges	Net Income
1st	$22,147,974	$3,661,198	$ 8,627,973	$1,591,523	$ 8,827,988	$ 500,708†
2nd	23,282,408	3,492,361	9,097,061	1,659,518	8,961,126	72,342
3rd	24,240,592	3,636,088	9,081,213	1,660,236	9,324,559	538,496
4th	23,961,398	3,594,209	8,825,665	1,808,951	9,531,232	201,341
5th	21,315,455	3,647,318	8,677,465	1,783,540	9,622,631	584,501
6th.	27,279,517	4,091,928	9,382,587	1,812,541	9,614,908	2,377,553
7th	29,726,927	4,459,039	10,723,912	2,106,769	9,573,522	2,863,585
8th	31,704,427	4,755,664	13,355,575	2,128,819	9,629,553	1,534,816
9th	36,039,520	4,955,124	17,287,117	2,345,750	9,735,652	1,715,877
10th	39,400,341	5,965,409	20,628,504	2,601,253	9,823,110	382,065
11th	42,911,040	8,560,400	19,874,369	2,798,821	9,870,158	1,807,292
12th	43,235,972	8,560,400	20,407,117	2,586,001	9,853,177	1,829,277
13th	45,552,031	8,560,400	22,479,553	2,695,708	10,016,370	1,800,000
14th	46,215,488	8,560,400	22,678,896	2,760,903	10,104,921	1,810,365

† Loss.

3. Prepare a ratio chart to show the following expenses:

Salaries	5 0%
Rent	8
Credit Losses	5
Heat	1
Light	1
Taxes	3
Shipping and Receiving	5
Depreciation	2 5
Miscellaneous Expenses	1 0
Power	5
Freight	5
Delivery Expense	1
Insurance	1

HINT. Rearrange items from highest to lowest per cent.

Index Numbers

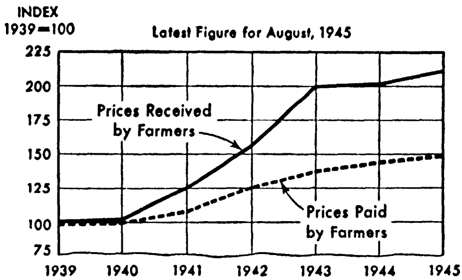
Uses of index numbers. Current newspapers, magazines, bulletins, and books contain much economic information expressed in terms of index numbers. Following are examples from the *Monthly Review of Agricultural and Business Conditions* issued by the Federal Reserve Bank:

"From June 1939 to March 1942 prices rose 30 per cent, according to the *index* of wholesale prices compiled by the U. S. Bureau of Labor Statistics."

"Industrial production reached a peak in February (1945) when the *index* stood at 236 per cent of the 1935–1939 average."

"The *index* of prices paid by farmers was unchanged at 173 for the fourth consecutive month according to data from the Department of Agriculture." (July 30, 1945)

Index numbers may be used in the form of a chart similar to the following, which appeared in a recent issue of the Chicago *Tribune:*



Index numbers. An index number is a number that is used for measuring trends in prices or in other movements which can be quantitatively expressed by means of statistical data. Among those more frequently used in business are the following:

Production indexes. Automobile production, building construction, electric power output, steel production (percentage of capacity).

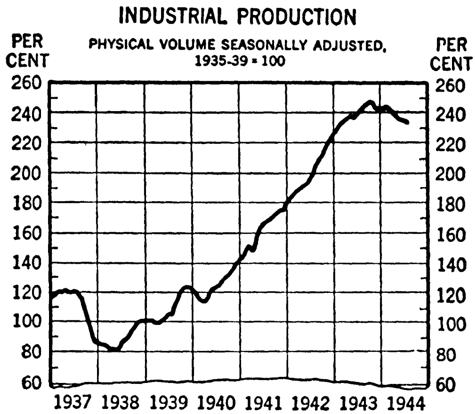
Trade indexes. Carloadings, department store sales, check clearings, inventories (of manufacturers, wholesalers, and retailers), value of imports and exports.

Financial indexes. Prices of stocks and bonds, business failures, new capital issues, commercial bank loans, prices of basic commodities, prices of agricultural products, foreign exchange rates.

Miscellaneous. Wage rates, employment, financial situation of the government, foreign affairs.

Most of the indexes listed in the foregoing are expressed statistically; therefore, their trends can be measured mathematically.

The nature of the particular business primarily determines the indexes most useful to it. To answer the question, "How is business?" recourse may be had for purposes of comparison to some index of general business conditions, for example, to the Federal Reserve index of industrial production, a national summary charted for a period of years:



or possibly to indexes somewhat more restrictive as to locality and type of business, such as the following from the *Ninth Federal Reserve District Monthly Review* of July 30, 1945:

NORTHWEST BUSINESS INDEXES
ADJUSTED FOR SEASONAL VARIATIONS—1935-1939 = 100

	June 1945	*May 1945*	*June 1944*	*June 1943*
Bank Debits—93 Cities	237	201	208	176
Bank Debits—Farming Centers	224	207	201	173
City Dept. Store Sales	183	172	153	146
City Dept. Store Stocks	173	163	152	137
Country Dept. Store Sales	162	150	149	143
Country Lumber Sales	110	100	102	128
Miscellaneous Carloadings	114	120	114	109
Total Carloadings (excl. Misc.)	139	203	143	145
Farm Prices—Minn. (unadj.)	185	180	177	178
Employment—Minn. (unadj. 1936 = 100)	......	134	143	151
Minnesota Payrolls (unadj. 1936 = 100)	......	229	233	228

Economic position of agriculture. Agriculture creates a stream of income that exceeds that of any other industry. This income flows through various channels into the industrial and commercial life of the nation. Probably as many or more people are engaged in the processing and handling of agricultural commodities from the producer to the consumer than are engaged in the actual production processes. Since agriculture and other industries are customers of one another, any enhancement of the buying power of one operates to the advantage of the other. With this background, the industry of agriculture is chosen to furnish much of the material in this chapter.

Suppose we desire to compare the price of wheat for different years. The July 15, 1937–1941 average price of wheat was

73 cents a bushel, and on July 15, 1945, the price was \$1.48 a bushel. We can say that wheat has gone up 75 cents a bushel, but a better comparison is to show what per cent the 1945 price is of the 1937–1941 average.

$$1.48 \div .73 = 2.03, \text{ approximately}$$
$$2.03 \times 100 = 203\%$$

Thus the 1945 price of wheat is 203% of the 1937–1941 average. Expressed as an index number, it is 203 when the base is "1937–1941 average price equals 100."

Construction of index numbers. In the study of price fluctuations, the first thing to determine is the original base price. This may be: (*a*) the price of a commodity on a certain date; (*b*) the average price of a commodity during a certain year; or (*c*) the average price of a commodity during several years. Whichever is used, that price is assigned a numerical value of 100.

Other index numbers for following years or periods are obtained by multiplying the price for the years or periods by 100 and dividing the product by the original base price. Thus, the index numbers of the prices paid to Minnesota wheat growers from June 15, 1939 to June 15, 1944, were as follows:

Average Price Received by Minnesota Farmers
(Fifteenth-of-Month Comparison)

All Wheat (bu.)	*Aver. Price per Bushel*	*Index Number or Relative Price*
1930–1939 average	\$ 79	100
1940	75	95
1941	85	108
1942	1 01	128
1943	1 27	161
1944 (6 mos.)	1 48	187

Index numbers or relative prices may be found for all farm products for which average prices are known. The base period taken for making calculations is arbitrarily selected, depending on the period for which comparisons are wanted.

Problems

Given the following data, compute the index numbers:

1. Corn (bu.):

1930–1939 average	\$ .52
1940	.48
1941	53
1942	68
1943	.90
1944 (6 mos.)	1.01

2. Potatoes (bu.):

1930–1939 average	$ .50
1940	.50
1941	.48
1942	.93
1943	1.27
1944 (6 mos.)	1.12

3. Hogs (100 lbs.):

1930–1939 average	$ 6.78
1940	5.26
1941	9.04
1942	13.10
1943	13.70
1944 (6 mos.)	13.25

4. If the average farm price of wheat used as the base was 85 cents a bushel (as in 1941), what is the index number for 1944, when the price was $1.48 cents a bushel?

Composite price indexes. It is commonly desired to express the price of a group of commodities or of all commodities in a single number or series. Thus, we say that the index of prices received by farmers in mid-April (1945) was 203, calculated from the average base price of 1910–1914. The index 203 was obtained by taking the average price for all farm products for each of the two periods of comparison. At the same time, the prices paid by farmers for things used in farm production and family maintenance, including interest and taxes, was 173, calculated from the average base price of 1910–1914. The ratio of the indexes of prices received to prices paid is the so-called *parity ratio* established by law which designates the years 1910–1914 as the base period. This ratio averaged 100 in the base years. It was 117 in mid-April (1945): $203 \div 173 = 1.17$, and $1.17 \times 100 = 117$. At the bottom of the depression in 1932, the parity ratio was 61. The term *parity*, as applied to the price of an agricultural commodity, is that price which will give to the commodity a purchasing power equivalent to the average purchasing power of the commodity in the base period, 1910–1914.

Weighted index numbers. An index number of prices which will satisfactorily measure changes in the price level must include a considerable number of representative commodities, and these commodities must be weighted in accordance with their importance in trade and industry. The number of items may be 30, or the number may be 500 or even more. For purposes of illustration, we shall use three principal agricultural commodities: wheat, corn, and cotton.

Suppose we desire to compute the average rise in the price of the three commodities from 1939 to 1942, making due allowance for the total quantity of each commodity produced, so that the index of prices found will give us information on the increase in buying power accruing to the producers as a result of the rise in prices. We obtain the production for the base period (1939) and the average prices for the years 1939 and 1942 from the *Yearbook* of the Department of Agriculture for 1943 and set up the material as follows:

	Wheat (*bu.*)	*Corn* (*bu.*)	*Cotton* (*500-lb. bales*)	*Total*
Total production in 1939	741,180,000	2,580,912,000	11,817,000	
Average price in 1939 . .	78¢ a bu.	56 8¢ a bu.	$45 45 a bale	
Average price in 1942.	$1 26 a bu.	85 5¢ a bu.	$94 65 a bale	
Value at 1939 prices	$578,120,400	$1,465,958,016	$ 537,082,650	$2,581,161,066
Value at 1942 prices	$933,886,800	$2,206,679,760	$1,118,509,050	$4,259,075,610

Price index: $4,259,075,610 ÷ $2,581,161,066 = 1.65
1.65 × 100% = 165%
Average rise in price: 165% − 100% = 65%

Explanation. The values of the commodities produced in 1939 were calculated at the average prices in both 1939 and 1942, and the total value of all three for each of the two years was found. The aggregate value for 1942 was then divided by the aggregate value for 1939. This method of constructing an index number of prices is recommended because it does not require the statistics of current production. (It is frequently impossible to obtain such statistics promptly.)

The calculation procedure of the foregoing example may be stated in terms of a formula for weighted aggregative price index as:

$$P = \frac{\Sigma p_n q_o}{\Sigma p_o q_o},$$

where:

p represents the price of a commodity;
q represents the quantity of a commodity;
o represents the base period, the period from which the price changes are measured;
n represents the given period, the period being compared with the base;
Σ is the symbol of summation, or addition.

In the illustration for 1942, the index number formula is:

$$P = \frac{\Sigma p_{1942} q_{1939}}{\Sigma p_{1939} q_{1939}}$$

There is no short cut. The q_o in the numerator and the q_o in the denominator may not be canceled. The quantity of each com-

modity included in the index must be multiplied by its respective price in both the base year and the given year and the several products added, as indicated in the formula and illustrated in the example.

Problems

1. Corn and barley are two principal feed grains used in fattening hogs for market. From the following figures calculate the weighted index for 1942, using 1939 as the base year.

	Hogs (lbs.)	*Corn (bu.)*	*Barley (bu.)*
Total production in 1939	17,081,824,000	2,580,912,000	278,163,000
Average price in 1939 (cents)......	7 74	56 8	40 5
Average price in 1942 (cents)......	13 04	85 5	59 4

2. Calculate the individual price indexes for hogs, corn, and barley. Did the price of hogs keep pace with the price of corn? How about the price of hogs and the price of barley? If both feeds were available, which would be the more profitable to use, assuming that each has the same feed value for pork production?

3. Find the weighted index for the three following fresh fruits sold in large quantities for home canning.

	Apples (bu.)	*Peaches (bu.)*	*Pears (bu.)*
Total production in 1939 .	139,379,000	64,222,000	29,279,000
Average price in 1939	$ .64	$ 82	$ 70
Average price in 1942	$1 38	$1 49	$1.57

4. In the following tabulation are given the index numbers for the average price paid by farmers and the average price received by farmers for the years 1932 to 1942, inclusive, based on 1910–1914 averages, the years chosen for computation of parity prices, at which time parity was 100.

	Prices Paid	*Prices Received*
1932	124	62
1933	120	81
1934	129	103
1935	130	107
1936	128	125
1937	134	105
1938	127	93
1939	125	96
1940	126	103
1941	133	142
1942	151	175

Calculate the parity price indexes for the years 1935, 1940, and 1942.

Farm evaluation on the basis of crop production index. The farm owner, the banker or financial agent making farm loans, and other interested persons may partially evaluate a farm in terms of crop production. Other factors, such as building improvements, fencing, drainage, location relative to roads, rural electrification line, and so forth, are also to be considered in fixing the total value.

The crop production index is a comparison of the yield per acre

of all crops on a given farm with the average yield per acre of all crops on a number of farms in the same locality—community, township, or county. A crop production index of 92 indicates that the yield of crops on a farm with this index is 92% of the average for the locality; a farm with a crop production index of 105 has a yield 5% greater than the average.

Computation of the crop production index. Assume that a farm has a cropping system as follows:

50 acres of corn yielding 40 bushels per acre
50 acres of oats yielding 35 bushels per acre
50 acres of alfalfa yielding $2\frac{1}{2}$ tons per acre

and that the average yields for the county in which this farm is located are:

Corn	30 bushels per acre
Oats	25 bushels per acre
Alfalfa	3 tons per acre

The crop production index is computed as follows:

$50 \times 40 = 2{,}000$, the number of bushels of corn produced
$50 \times 35 = 1{,}750$, the number of bushels of oats produced
$50 \times 2\frac{1}{2} = 125$, the number of tons of alfalfa produced
150 acres

Next, find how many acres would be required to produce each crop using the county averages.

2,000 bu. of corn ÷ 30 bu. per acre would require	$66\frac{2}{3}$ acres for corn
1,750 bu. of oats ÷ 25 bu. per acre would require	70 acres for oats
125 tons of alfalfa ÷ 3 tons per acre would require	$41\frac{2}{3}$ acres
	$178\frac{1}{3}$ acres required

Since $178\frac{1}{3}$ acres would be needed to produce what this farm produced, on the basis of county averages, the crop production index is 118.9.

$$\frac{178\frac{1}{3}}{150} \times 100 = 118.9$$

Problem

Ten acres of a 160-acre farm were used for farmstead and pasture, and the 150 acres in cultivation produced the following crops, as compared with the county average:

Crop	*Number of Acres*	*Yield per Acre*	*County Average Yield per Acre*
Corn	40	45 bu.	$42\frac{1}{2}$ bu.
Oats	40	50 bu.	48 bu.
Alfalfa.	30	2 tons	$2\frac{1}{2}$ tons
Wheat.	40	25 bu.	20 bu.
	150		

Compute the crop production index for this farm.

CHAPTER 28

Progression

Definition. A progression is a series of numbers, each term of which is obtained from the preceding or following term by a fixed law; as, 2, 4, 6, 8, and so forth; or, 2, 4, 8, 16, and so forth.

Increasing series. A progression, each term of which is greater than the preceding term, is known as an increasing or ascending series.

Decreasing series. A progression, each term of which is less than the preceding term, is known as a decreasing or descending series.

Arithmetical progression. When each term of a progression differs from the preceding or following term by a common difference, the progression is said to be arithmetical.

Symbols. The five elements of an arithmetical progression are represented by certain well-established symbols:

Number of terms	n
First term	a
Last term	l
Common difference	d
Sum of the terms	s

Relation of elements. The five elements whose symbols are indicated above are so related that if any three of them are given, the remaining two may be found.

Increasing Series

The formulas used in connection with increasing series, and the methods of solution, are shown below; in each case the values of the terms have been taken as follows:

Number of terms	12
First term	2
Last term	35
Common difference	3
Sum	222

To find the number of terms.

Algebraic Formula	*Arithmetical Substitution*
$\frac{l - a}{d} + 1 = n.$	$\frac{35 - 2}{3} + 1 =$ number of terms.

Solution

Simplifying the numerator: $35 - 2 = 33$
Dividing by denominator: $33 \div 3 = 11$
Adding: $11 + 1 = 12$, the number of terms

To find the first term.

Algebraic Formula	*Arithmetical Substitution*
$l - (n - 1)d = a.$	$35 - (12 - 1)3 =$ first term.

Solution

Removing parentheses: $12 - 1 = 11$
Multiplying by 3: $11 \times 3 = 33$
Subtracting: $35 - 33 = 2$, the first term

To find the last term.

Algebraic Formula	*Arithmetical Substitution*
$a + (n - 1)d = l.$	$2 + (12 - 1)3 =$ last term.

Solution

Removing parentheses: $12 - 1 = 11$
Multiplying by 3: $11 \times 3 = 33$
Adding: $2 + 33 = 35$, the last term

To find the common difference.

Algebraic Formula	*Arithmetical Substitution*
$\frac{l - a}{n - 1} = d.$	$\frac{35 - 2}{12 - 1} =$ common difference.

Solution

Simplifying the numerator: $35 - 2 = 33$
Simplifying the denominator: $12 - 1 = 11$
Dividing: $33 \div 11 = 3$, the common difference

To find the sum.

Algebraic Formula	*Arithmetical Substitution*
$\frac{n}{2}(a + l) = s.$	$\frac{12}{2}(2 + 35) =$ sum.

Solution

Adding the terms in parentheses: $2 + 35 = 37$

Multiplying by the fraction: $37 \times \frac{12}{2} = 222$, the sum

Decreasing Series

The decreasing series will be illustrated in the same manner as the increasing series, with the values of the terms taken as follows:

Number of terms	8
First term	7
Last term	−21
Common difference	−4
Sum	−56

To find the number of terms.

Algebraic Formula — *Arithmetical Substitution*

$$\frac{l-a}{d}+1=n. \qquad \frac{(-21)-7}{-4}+1=\text{number of terms.}$$

Solution

Simplifying the numerator: $(-21) - 7 = -28$
Dividing the denominator: $(-28) \div (-4) = 7$
Adding: $7 + 1 = 8$, the number of terms

To find the first term.

Algebraic Formula — *Arithmetical Substitution*

$$l-(n-1)d=a. \qquad -21-[(8-1)(-4)]=\text{first term.}$$

Solution

Removing the parentheses: $8 - 1 = 7$
Multiplying: $7 \times (-4) = -28$
Simplifying: $(-21) - (-28) = (-21) + 28$
And: $-21 + 28 = 7$

To find the last term.

Algebraic Formula — *Arithmetical Substitution*

$$a+(n-1)d=l. \qquad 7+[(8-1)(-4)]=\text{last term.}$$

Solution

Removing parentheses: $8 - 1 = 7$
Multiplying by -4: $7 \times (-4) = -28$
Subtracting: $7 - 28 = -21$

To find the common difference.

Algebraic Formula — *Arithmetical Substitution*

$$\frac{l-a}{n-1}=d. \qquad \frac{(-21)-7}{8-1}=\text{common difference.}$$

Solution

Simplifying the numerator: $(-21) - 7 = -28$
Simplifying the denominator: $8 - 1 = 7$
Dividing: $-28 \div +7 = -4$

To find the sum.

Algebraic Formula — *Arithmetical Substitution*

$$\frac{n}{2}(a+l)=s. \qquad \frac{8}{2}\Big[7+(-21)\Big]=\text{sum.}$$

Solution

Simplifying the fraction: $\frac{8}{2} = 4$

Adding: $7 + (-21) = -14$

Multiplying: $4 \times (-14) = -56$

Problems

1. Given $a = 2$, $n = 6$, $l = 12$; find d and s.

2. Find the sum of all the even numbers from 10 to 80 inclusive.

3. The first term of a progression is 6, the number of terms is 15, the common difference is 7; find the last term and the sum.

4. Given $n = 12$, $l = -17$, $s = -72$, find d and a.

5. The first term is 6, the last term is 181, and the common difference is 7; find the number of terms.

6. Given $l = 57$, $n = 23$, $a = -9$; find d and s.

7. A building is to be leased for a term of 21 years. The first year's rental is to be $10,000.00, equal increases in rent are to be made each year, and the rental for the twenty-first year is to be $30,000.00. Find: (*a*) the difference in each year's rental; (*b*) the total rental that will be paid during the period of 21 years.

8. *A* deposited $25.00 in his savings account on January 1, and on the first of each month thereafter deposited $5.00 more than the previous month. How many dollars did he deposit December 1, and what was the amount of the accumulated deposits? Do not take interest into consideration in solving this problem.

9. A punch board has 50 numbers in each section (numbers 1 to 50). A person pays the amount of the number punched. If the board has four sections, what will be the amount of revenue derived from the board?

10. A man invests his savings in the shares of a building and loan association, depositing $240.00 the first year. At the beginning of the second year he is credited with $16.80 interest, and deposits $223.20. At the beginning of the third year he is credited with $33.60 interest, and deposits $206.40. What is his credit at the end of 10 years, and how much cash has he paid in?

11. A bond issue of $40,000.00 bearing interest at 4% is to be retired in 10 equal annual installments. What amount of interest will be paid during the life of the issue?

12. An employee started work for a company at $1,200.00 for the first year, with a guaranteed yearly increase of $100.00. What was his salary 12 years later? How much had the company paid him in the course of the 12 years?

Geometrical Progression

A geometrical progression is one in which any term after the first is obtained by multiplying the preceding term by a fixed number known as the ratio.

Elements. In a geometrical progression, there are five elements so related that any three being given, the others may be found. These five elements, together with their symbols, are:

Number of terms	n
First term	a
Last term	l
Sum of the series	s
Ratio	r

Increasing series. As in an arithmetical progression, the formulas and solutions in a geometrical progression are based on one set of terms, and the solutions are given as though the required term in the example in question were lacking. In some cases, two different formulas may be used.

Number of terms	6
First term	3
Last term	96
Sum of terms	189
Ratio of increase	2

To find the first term.

Algebraic Formula *Arithmetical Substitution*

$$\frac{l}{(r)^{n-1}} = a. \qquad \frac{96}{(2)^{6-1}} = \text{first term.}$$

Solution

Simplifying the exponent: $(2)^{6-1} = (2)^5$
Finding the power: $(2)^5 = 32$
Dividing: $96 \div 32 = 3$, the first term

To find the last term.

Algebraic Formula *Arithmetical Substitution*

$$a(r)^{n-1} = l. \qquad 3(2)^{6-1} = \text{last term.}$$

Solution

Simplifying the exponent: $(2)^{6-1} = (2)^5$
Finding the power: $(2)^5 = 32$
Simplifying: $3 \times 32 = 96$, the last term

To find the sum.

Algebraic Formula *Arithmetical Substitution*

$$\frac{lr - a}{r - 1} = s. \qquad \frac{(96 \times 2) - 3}{2 - 1} = \text{sum.}$$

Solution

Simplifying: $(96 \times 2) = 192$
Subtracting: $192 - 3 = 189$
Simplifying the denominator: $2 - 1 = 1$
Dividing: $189 \div 1 = 189$, the sum

To find the ratio.

Algebraic Formula	*Arithmetical Substitution*
$\sqrt[n-1]{\frac{l}{a}} = r.$	$\sqrt[6-1]{\frac{96}{3}} = \text{ratio.}$

Solution

Simplifying the exponent: $6 - 1 = 5$
Simplifying the fraction: $96 \div 3 = 32$
Extracting the 5th root: $\sqrt[5]{32} = 2$, the ratio

Decreasing series.

Number of terms	6
First term	96
Last term	3
Ratio of decrease	$\frac{1}{2}$
Sum of terms	189

To find the first term.

Algebraic Formula	*Arithmetical Substitution*
$\frac{l}{(r)^{n-1}} = a.$	$\frac{3}{(\frac{1}{2})^{6-1}} = \text{first term.}$

Solution

Simplifying the denominator: $(\frac{1}{2})^{6-1} = \frac{1}{32}$
Dividing: $3 \div \frac{1}{32} = 96$, the first term

To find the last term.

Algebraic Formula	*Arithmetical Substitution*
$a(r)^{n-1} = l.$	$96(\frac{1}{2})^{6-1} = \text{last term.}$

Solution

Simplifying the exponents: $6 - 1 = 5$
Finding the power: $(\frac{1}{2})^5 = \frac{1}{32}$
Multiplying: $96 \times \frac{1}{32} = 3$, the last term

To find the sum.

Algebraic Formula	*Arithmetical Substitution*
$\frac{lr - a}{r - 1} = s.$	$\frac{(3 \times \frac{1}{2}) - 96}{\frac{1}{2} - 1} = \text{sum.}$

Solution

Multiplying: $3 \times \frac{1}{2} = \frac{3}{2}$
Subtracting: $\frac{3}{2} - 96 = -\frac{189}{2}$
Simplifying the denominator: $\frac{1}{2} - 1 = -\frac{1}{2}$
Dividing: $-\frac{189}{2} \div -\frac{1}{2} = 189$, the sum

To find the ratio.

Algebraic Formula	*Arithmetical Substitution*
$\sqrt[n-1]{\frac{l}{a}} = r.$	$\sqrt[6-1]{\frac{3}{96}} = \text{ratio.}$

Solution

Simplifying the fraction: $\frac{3}{96} = \frac{1}{32}$

Simplifying the radical: $\sqrt[5]{\frac{1}{32}} = \frac{1}{2}$, the ratio

Progression problems solved by the use of logarithms. The use of logarithms replaces laborious calculations in solving problems in progression.

Example

What is the average yearly rate of increase if \$100.00 placed at interest for 10 years produces \$179.08?

Algebraic Formula

$$R = \sqrt[n]{\frac{\text{Value at end}}{\text{Value at beginning}}} - 1.$$

Arithmetical Substitution

$$\text{Rate} = \sqrt[10]{\frac{179.08}{100.00}} - 1.$$

Solution

$$179.08 \div 100.00 = 1.7908$$
$$\log 1.7908 = 0.253047$$
$$\log 0.253047 \div 10 = \log 0.025305$$

In the table of logarithms, it is found that log 0.025305 stands for 1.06.

$$1.06 - 1.00 = .06, \text{ or } 6\%$$

Problems

1. A machine costing \$27,500.00 is found to be worth only \$2,750.00 at the end of 12 years. Find the fixed percentage of diminishing value.

2. A building cost \$80,000.00. At the end of each year the owners deducted 10% from its carrying value as estimated at the beginning of the year. What is the estimated value at the end of 10 years? NOTE: The value at the end of the 10th year is the value at the beginning of the 11th year; therefore: Value = \$80,000 $\times$ $(.9)^{11-1}$.

3. An asset that cost \$15,000.00 has been written down 3% of the decreasing balance each year for 10 years. At the end of the 10th year, what is its value as shown by the books?

4. If the population of a city increases in 5 years from 150,000 to 175,000, find the average rate of yearly increase.

5. If a savings bank pays 5% compounded annually, what will be the amount of \$200.00 at the end of 6 years?

6. There are seven terms in a geometric progression of which the first term is 2 and the last term is 1.458. Find r and s.

7. Find a and s in a geometric progression where $r = 2$, $n = 9$, and $l = 256$.

8. Find r and s in a geometric progression where $a = 17$, $l = 459$, and $n = 4$.

9. If the bacteria in milk double every two hours, how many times will the number be multiplied in 24 hours?

CHAPTER 29

Foreign Exchange

Foreign trade. The foreign trade connections of American concerns make it necessary for the accountant to be acquainted with the basic principles of exchange values.

Rate of exchange. Theoretically, the rate of exchange between any two countries is the ratio between the values of the amounts of metal in their standard monetary units. This theoretical rate is commonly known as "par of exchange," but because of the many economic factors in foreign business, another rate, the "current rate," is usually used.

Par of exchange. A country that stands ready to redeem all of its obligations in gold, upon demand and without restriction, is said to be "on the gold standard." Conditions at this writing (April 1946) are chaotic, and all countries are "off the gold standard." However, coinages of world monetary systems are based on actual or theoretical monetary units containing gold or silver of a legally established weight and fineness. The mint par rate of exchange between any two countries is the ratio between the amounts of gold, if the countries are on a gold basis, or of silver, if they are on a silver basis, contained in their standard monetary units. If one country is on a gold basis and another is on a silver basis, reference must be made to the market prices of the two kinds of bullion. If a foreign coin contains 516.4058 grains of fine silver, and the market price of silver in terms of gold in this country is 60 cents for 5 ounces (480 grains), the rate of exchange is: $(516.4058 \div 480) \times .60 = .646$. The silver par of exchange is not fixed because the market price of silver in terms of gold money is continually changing. Calculation of the par rate of exchange of the English sovereign and the United States gold dollar is shown in the following example.

Example

Find the par of exchange of the English sovereign and the United States gold dollar.

Solution

Weight of English sovereign piece	123 2744700 grains
Deduct alloy, $8\frac{1}{3}\%$	10 2728725
Net weight of gold	113 0015975
*Weight of United States gold dollar	15 2380952 grains
Deduct alloy, 10%	1 5238095
Net weight of gold	13 7142857

The par of exchange is $113.0015975 \div 13.7142857 = 8\ 2396998$

A comparative table showing the values of foreign coins is issued at frequent intervals by the United States Treasury Department.

Current rate of exchange. The current rate of exchange, under normal conditions, will fluctuate between the gold import and the gold export point, slightly below or above par. This fluctuation is caused chiefly by demand and supply.

Under abnormal conditions, the rates are quite different from par, owing to a number of causes, some of which are:

(1) Suspension of the gold standard in the country where the rate is quoted, or in the country whose currency is quoted.

(2) The extent to which the currency of either country is depreciated as a result of management of the currency or inflation.

(3) Lack of confidence in the stability of foreign government.

(4) Issuance of a large amount of paper money by the foreign country.

(5) Shrinkage of gold and other legal reserves.

Sterling quotations are usually made on "demand," but may be made on 30-, 60-, or 90-day drafts. Where time is a factor in calculations, three days of grace are allowed, interest being calculated on the basis of 365 days to the year for sterling drafts.

Six classes of problems. The mathematics of foreign exchange may be resolved into six classes of problems:

(1) Conversion of one monetary unit into terms of another.

(2) Interest on foreign exchange.

(3) Bankers' and brokers' problems of valuing time bills of exchange.

(4) Finding the value of an account as a whole.

(5) Averaging accounts of foreign exchange bearing interest.

(6) Foreign branch house accounting.

Conversion of one monetary unit into terms of another. The necessity of converting one monetary unit into terms of another

* By proclamation of the President, the weight of the gold dollar was fixed at $15\frac{5}{21}$ grains nine-tenths fine on January 31, 1934.

results from the simple purchase of letters of credit, travelers' checks, and cable transfers on a foreign country. These transfers are calculated at the quoted or current rate of exchange. This current rate is the rate prevailing for that particular class of transfer on the date of purchase.

To make the calculations, reduce the amount of foreign money to the term of the monetary unit to which the exchange rate is applicable, that is, in units and a decimal fraction thereof, as, £150 7s. 6d. = £150.375. Multiply by the conversion rate and add the charges for the transfer.

Example

What is the cost of a letter of credit on London for £150 7s. 6d., if the bank charges $\frac{1}{2}$% for its service? Assume that the current rate of exchange is $4.05.

Solution

	£150 000
7s. = $\frac{7}{20}$ of £1	350
6d. = $\frac{6}{240}$ of £1	025
	£150 375
Multiply by rate of exchange	4 05
Dollar value at current rate	609 02
Service charge of $\frac{1}{2}$%	3 05
Total cost	612 07

Conversion of decimals of one monetary unit into monetary units of a smaller denomination. This is simply a matter of reduction of decimals of denominate numbers.

Example

What value of London exchange may be purchased with $1,000, if the rate is 4.46\frac{3}{8}$?

Solution

$1,000 ÷ $4 46375 = 224 0269, the number of pounds sterling
.0269 × 20 = .538s.; therefore, no shillings
.538 × 12 = 6.456d.

Answer: £224 0s. 6d.

Problems

Find the cost in dollars of each of the following:

1. £1,200 16s. at 3.75\frac{1}{2}$.
2. £8,240 5s. 8d. at $3.90; commission, 1%.
3. Calculate how much exchange can be purchased with the following. $750 on England, rate, $3.845.

Interest on foreign exchange. To find the interest on foreign exchange:

(*a*) Reduce the amount of foreign money to the term of the monetary unit to which the exchange rate is applicable.

(*b*) Compute the interest in the terms of that monetary unit.

(*c*) Reduce the interest to its proper exchange units.

Example

Find the interest on £200 5s. 6d. for 63 days at 6%.

Solution

£		£200.00
5s. = $\frac{1}{4}$ of £1 (decimally) .		.25
6d. = $\frac{1}{40}$ of £1 (decimally) .		.025
		£200.275
Interest for 60 days (point off 2 places) .		£ 2.0027500
Interest for 3 days ($\frac{1}{20}$ of 60 days' int.) .		.1001375
		£ 2.1028875
The above amount is ordinary interest and is changed to exact interest by deducting $\frac{1}{73}$		.0288066
		£ 2.0740809

Reducing the decimal of a pound to shillings, .0740809 × 20 = 1.481618 shillings. Reducing the decimal of a shilling to pence, .481618 × 12 = 5.779416, or practically 6 pence. Answer: £2 1s. 6d.

Problems

1. Find the interest on £250 10s. 6d. for 93 days at 6%.

2. Find the interest on £1,000 15s. 10d. for 63 days at 5%.

To find the value of a time bill of exchange. The four elements entering into the determination of the value of a time bill of exchange are:

(1) Current rate of demand exchange.
(2) Interest on money (exact time).
(3) Stamp charges.
(4) Commission of home or foreign banks.

Example

If the demand rate of exchange is $4.05000, discount rate 4%, stamps $\frac{1}{20}$%, and commission $\frac{1}{4}$%, what is the 60-day rate?

Solution

Demand rate		$4.05000
Deduct:		
Interest at 4% for $\frac{63}{365}$ of a year	.02796	
Stamps, $\frac{1}{20}$%	.00203	
Commission, $\frac{1}{4}$% . . .	.01013	
		.04012
60-day rate		$4.00988

Problem

Determine the proceeds of a 60-day draft on London for £200, if the following conditions prevail:

Demand rate on London	\$3.90
Interest rate	7%
Stamp charges	$\frac{1}{20}$%
Commission charged by English bank	$\frac{1}{4}$%
Commission charged by American bank	$\frac{1}{2}$%

Foreign exchange accounts. Rates of exchange fluctuate; therefore, it is necessary to establish a method of keeping records of transactions to show current values.

Procedure: (*a*) Extend the items of debits and credits in foreign currency at the rates of exchange stated for each item. (In practice, this would be performed when the entry is recorded.)

(*b*) Strike a balance of the foreign currency, and convert this balance at the current rate of exchange. Insert this result among the dollar items.

(*c*) Find the difference between the total debit dollars and the total credit dollars. A debit difference indicates a loss on exchange, while a credit difference indicates a profit on exchange.

Example

State the balance of the following account in both foreign and domestic currency, and show the profit or loss on exchange, exclusive of interest charges and credits. The current price of exchange is quoted at \$4.51 on the last day of the month.

DAVIS & COMPANY--BANKERS, LONDON

Debits:
- Jan. 1 Remittance, sight bill, £1,000 at \$4.46$\frac{3}{8}$
- Jan. 10 Remittance, sight bill, £500 10s. at \$4.49$\frac{1}{2}$

Credits:
- Jan. 8 Drafts drawn, £100 at \$4.47
- Jan. 24 Cable, £1,100 10s. 6d. at \$4.52$\frac{1}{2}$

Solution

		Foreign	*Rate*	*Domestic*
Debits:				
Jan. 1	Remittance	£1,000 0s.	\$4 46$\frac{3}{8}$	\$4,463 75
Jan. 10	Remittance	500 10s.	4 495	2,249 75
	Profit on exchange			66 27
		£1,500 10s.		\$6,779 77
Credits:				
Jan. 8	Drafts	£ 100 0s. 0d.	\$4 47	\$ 447 00
Jan. 24	Cable	1,100 10s. 6d.	4 525	4,979 88
	Balance	299 19s. 6d.	4 51	1,352 89
		£1,500 10s. 0d.		\$6,779 77

Explanation. The balance of the account is found by first deducting the smaller side of the account in foreign money from the larger side. In the above case, the smaller side amounts to £1,200 10s. 6d., and the larger side to £1,500 10s.; the balance is therefore £299 19s. 6d. Second, convert this balance into domestic money at the current rate of exchange, $4.51; the balance is found to be $1,352.89.

The profit or loss on exchange is the difference between the total debit and the total credit of the domestic money columns.

There might be a question, in determining the correct valuation of the balance on hand, as to whether the £299 19s. 6d. should have been valued at the last buying price of $4.49½, or at $4.51, the current market price. This is purely a question of accounting and finance. The usual practice is to take the current rate on the last day of the month, when given; otherwise, the cost of the exchange on hand, if that can be determined by inspection, will be acceptable.

Problems

1. Find the balance of the following account in both foreign and domestic currency, and the profit or loss on exchange, exclusive of interest:

Debits:					
Aug. 1 Balance	£ 500	10s.	4d.	@	$4.85
Aug. 10 Remittance	11,000	0s.	0d.	@	3 90
Aug. 20 Collection	200	0s.	6d.	@	3 875
Aug. 25 Discounts	196	5s.	6d.	@	3 88
Credits:					
Aug. 3 Cable	£ 200	0s.	0d.	@	$3 88
Aug. 7 Sight draft	400	10s.	0d.	@	3 875
Aug. 9 Sight draft	300	0s.	0d.	@	3 89
Aug. 15 Cable	2,000	15s.	6d.	@	3 87½
Aug. 20 Cable	400	0s.	0d.	@	3 87¼

Demand rate on Aug. 31, $3.90.

2.* A dealer in foreign exchange finds from his books that he has had the following transactions in London exchange during a particular month:

Exchange bought in local market:
Jan. 1 30-day bill, payable in London, £300 @ $4.75
Jan. 15 Bill due in London, at sight, £2,500 @ $4.76
Exchange sold in local market:
Jan. 5 Bill due in London, at sight, £1,000 @ $4.77
Jan. 20 Cable transfer, £2,000 @ $4.78
Foreign correspondent's draft honored and paid:
Jan. 20 Bill at 30 days after sight, accepted December 21, £500 @ $4.78

State how the balance of the account stands at the close of the month, and how much profit or loss has been derived from the transactions. (At January 31, the rate for cable transfers is $4.80.)

Is the profit or loss so stated final?

Averaging accounts in foreign exchange. In general, transactions in foreign exchange accounts are large, and involve the

* American Institute Examination.

holding of considerable sums of money; because of this, interest is a very important factor. If the interest has not been taken care of at the time of the transaction, it must necessarily be considered later.

The principle of averaging accounts, as explained previously in this text, will be applied here.

Example

Find the average due date of the following account; also the amount due June 1 following, including interest at 6%:

Debits		*Credits*	
Jan. 2	£600	Feb. 15	£500
Jan. 31	300	Apr. 1	200
Mar. 16	100		

Solution

	Debits	*Days*	*Day-£*	*Total*
Jan. 2	£ 600	2	£ 1,200	
Jan. 31	300	31	9,300	
Mar. 16	100	75	7,500	
	£1,000			£18,000
	Credits	*Days*	*Day-£*	*Total*
Feb. 15	£ 500	46	£23,000	
Apr. 1	200	91	18,200	
	£ 700			£41,200
Balance, £	300			£23,200

Using the last day of the month preceding the first item as the focal date, 23,200 ÷ 300 = 77 days. As the balances of the £'s and the Day-£'s are on opposite sides, the due date will be counted backward from the focal date; that is, £300 should have been paid 77 days before December 31, or on October 15, in order that neither party should lose interest.

From October 15 of one year to June 1 of the next year is 229 days. £300 for 229 days at 6% is:

$$300 \times .06 \times \frac{229}{365} = £11.29315$$

$$.29315 \times 20 = 5.8630\text{s.}$$
$$.863 \times 12 = 10.356\text{d.}$$

Answer:

Principal due October 15 last	£300 0s. 0d.
Interest to June 1	11 5s. 10d.
Total due June 1	£311 5s. 10d.

Problems

1. Find the average due date of the following account, and the amount due on July 1, including interest at 5%:

Debits		*Credits*	
June 1	£200 0s. 0d.	June 4	£400 15s. 9d.
June 4	300 10s. 6d.	June 12	400 0s. 0d.
June 6	400 0s. 0d.	June 24	500 15s. 0d.
June 12	600 15s. 3d.	June 30	600 0s. 0d.
June 25	700 0s. 0d.		
June 28	500 0s. 0d.		

2. Bond, who is located in New York, has an account with Waite in London. Waite engages an accountant to prepare from the following data a statement to be mailed to Bond:

Debits		*Credits*	
May 12	£ 650	June 10	£ 400
May 30	217	June 30	400
June 12	240	July 1, Balance	557
July 1	250		
	£1,357		£1,357

Find the average due date of the account, and the interest at 5% to July 1, using 365 days to the year.

Conversion of foreign branch accounts.* The conversion of foreign branch accounts requires the application of different rates of exchange.

Current asset and current liability values should be converted at the rate prevailing as of the date of the balance sheet.

Fixed asset values should be converted at the rate prevailing at the time of purchase, or at the average rate for purchases during a fiscal period. If there have been no changes in fixed assets during the fiscal period, the fixed assets should be valued at the same rate as in the preceding period. Differences in values of fixed assets from one period to another, due to fluctuations in exchange, should not be allowed to affect the results of the fiscal period.

Remittances should be converted at the actual rate prevailing on the day of the transmittal of the money.

Revenue items should be converted at the average rate for the period.

The Controlling or Adjustment account and the Old Inventory account should be converted at the rate established on the head office books at the end of the preceding fiscal period.

The following example illustrates the above principles of conversion.

Example

A corporation having its head office in Boston had a branch office in London. The trial balances of the head office and of the branch office on December 31 were:

* See page 294, "Current rate of exchange."

BOSTON OFFICE

Trial Balance

Cash	$ 100,000	
Branch Account	876,000	
Remittances		$ 130,500
Expenses	40,000	
Income		60,000
Capital Stock		800,000
Surplus		25,500
	$1,016,000	$1,016,000

LONDON BRANCH OFFICE

Trial Balance

Cash	£ 15,000	
Remittances	30,000	
Customers	130,000	
Inventory, December 31	50,000	
Expenses	10,000	
Income		£ 25,000
Creditors		10,000
Boston Office Control		200,000
	£235,000	£235,000

An analysis of the Remittance account showed:

June 1, from London Branch	£10,000 @ $4 42
Sept. 1, from London Branch	10,000 @ 4 28
Dec. 1, from London Branch	10,000 @ 4 35
Current rate	$4 40
Average rate	4 30

From the foregoing data, prepare:

(*a*) Statement of conversion.
(*b*) Consolidated trial balance working paper.
(*c*) Consolidated profit and loss statement.
(*d*) Consolidated balance sheet.

Solution

BRANCH OFFICE

Statement of Conversion

	Pounds	*Pounds*	*Rate*	*Dollars*	*Dollars*
Cash	15,000		$4 40	66,000	
Remittances	30,000		4 35	130,500	
Customers	130,000		4 40	572,000	
Inventory	50,000		4 40	220,000	
Expenses	10,000		4 30	43,000	
Income		25,000	4 30		107,500
Creditors		10,000	4 40		44,000
Boston Control		200,000	4 38		876,000
Profit on Exchange					4.000
	235,000	235,000		1,031,500	1,031,500

Consolidated Trial Balance Work Sheet

	Boston Office		*London Branch*		*Eliminations*		*Consolidated Trial Balance*	
Cash	$ 100,000		$ 66,000				$ 166,000	
Branch Account	876,000					$ 876,000		
Remittances from Branch		$ 130,500			$ 130,500			
Expenses at Home Office	40,000						40,000	
Income at Home Office		60,000						$ 60,000
Capital Stock		800,000						800,000
Surplus		25,500						25,500
Remittances to Home Office			130,500			130,500		
Customers			572,000				572,000	
Inventory			220,000				220,000	
Expenses at Branch			43,000				43,000	
Income at Branch				107,500				107,500
Creditors				44,000				44,000
Boston Control				876,000	876,000			
Profit on Exchange				4,000				4,000
	$1,016,000	$1,016,000	$1,031,500	$1,031,500	$1,006,500	$1,006,500	$1,041,000	$1,041,000

CONSOLIDATED PROFIT AND LOSS STATEMENT

Income:		
Boston Office	$ 60,000	
London Office	107,500	
Total		$167,500
Expenses:		
Boston Office	$ 40,000	
London Office	43,000	
Total		83,000
Operating Income		$ 84,500
Profit on Exchange†		4,000
Net Profit to Surplus		$ 88,500

† Some may question the advisability of including the Profit on Exchange in the Net Profits. The setting up of an account "Reserve for Fluctuation of Exchange" is sometimes advocated. This is purely an accounting problem, however, and will not be discussed here.

CONSOLIDATED BALANCE SHEET

Assets

Cash:			
Boston		$100,000	
London		66,000	$166,000
Customers			572,000
Inventory			220,000
			$958,000

Liabilities and Capital

Creditors			$ 44,000
Capital:			
Capital Stock		$800,000	
Surplus:			
Balance, Jan. 1	$25,500		
Net Profit	88,500	114,000	914,000
			$958,000

In the above example, no fixed assets were stated. When, as sometimes happens, this item appears in the Branch Trial Balance, other difficulties are encountered.

To overcome these difficulties, it is necessary, in making a statement of conversion, to divide the account on the Head Office books, "Branch Control," into "Branch Control—Fixed Assets," and "Branch Control—Current Assets." The rate at the time of purchase should be used for the conversion of the Branch Control—Fixed Assets account. By converting at this rate, the value of the fixed assets is found in terms of the monetary unit of the country in which the Head Office is located. By deducting the value of the fixed assets, or the Branch Control—Fixed Assets, from the Branch Control account, the value of the Branch Control—Current Assets is found.

The following example illustrates this point:

Example

A United States company doing business in London through its London Branch, received a trial balance of the branch on December 31, as follows:

LONDON BRANCH

Cash	£ 5,000	
Remittances	25,000	
Customers	50,000	
Inventory (new)	25,000	
Expenses	25,000	
Plant	100,000	
Creditors		£ 50,000
Income from Sales		50,000
New York Control		130,000
	£230,000	£230,000

NEW YORK OFFICE

Cash	$100,000	
London Branch	617,500	
Remittances		$118,875
Expenses	50,125	
Capital Stock		500,000
Surplus		148,750
	$767,625	$767,625

The following remittances were received:

£5,000 @ $4.75 £5,000 @ $4.75¼
£5,000 @ $4.75½ £5,000 @ $4.75¾
£5,000 @ $4.76

The current rate of exchange was $4.76½.
The average rate of exchange was $4.75¼.
The fixed rate of exchange for the plant was $4.74½.

From the information given, show: (*a*) a statement of conversion; (*b*) a balance sheet of the London Branch; and (*c*) a consolidated balance sheet of the New York Office. Show also: (*d*) the accounts on the books of the New York Office in which the profit is taken up.

Solution

STATEMENT OF CONVERSION

Cash	£ 5,000		@ $4 76½	$ 23,825 00	
Remittances	25,000		@ 4 75½	118,875 00	
Customers	50,000		@ 4 76½	238,250 00	
Inventory	25,000		@ 4 76½	119,125 00	
Expenses	25,000		@ 4 75¼	118,812 50	
Plant	100,000		@ 4 74½	474,500 00	
Creditors		£ 50,000	@ 4 76½		$ 238,250 00
Income from Mdse		50,000	@ 4 75¼		237,625 00
New York Control (Plant Acct.)		100,000	@ 4 74½		474,500 00
New York Control (Current Acct.)		30,000	@ 4.76⅔		143,000 00
Profit on Exchange					12 50
	£230,000	£230,000		$1,093,387 50	$1,093,387 50

Conversion of Branch Control Account

Plant and Fixed Assets

London Branch Control		$617,500
Deduct:		
Value of Plant	£100,000	
Rate of exchange for fixed assets	4.74½	
Value in dollars		474,500
		$143,000
New York Control	£130,000	
Less Plant Account	100,000	
Current Assets at Branch	£ 30,000	
		143,000

The rate is found to be 143,000 ÷ 30,000 = 4.76⅔.

Balance Sheet of London Branch

Assets				
Cash . .	£ 5,000	@ $4 76½		$ 23,825
Customers	50,000	@ 4.76½		238,250
Inventory	25,000	@ 4.76½		119,125
Plant	100,000	@ 4.74½		474,500
	£180,000			$855,700
Liabilities				
Creditors	50,000	@ $4.76½		238,250
Capital	£130,000			$617,450
Plant	100,000	@ $4 74½	$474,500	
Current (Schedule A)	30,000	@ 4 76½	142,950	617,450

Schedule A

Current Account

Credits				
Balance	£30,000	@ $4 76⅔	$143,000 00	
Income	50,000	@ 4 75¼	237,625.00	
	£80,000			
Profit on Exchange			12.50	
				$380,637.50
Debits				
Expenses	£25,000	@ $4 75¼	$118,812 50	
Remittances	25,000	@ 4.75½	118,875 00	
	£50,000			237,687 50
Balance	£30,000			$142,950 00

The rate is found to be 142,950 ÷ 30,000 = 4.76½.

NEW YORK OFFICE

London Control Account

Bal. Plant Acct ..	$474,500 00	Remittances............	$118,875.00
Bal. Current Acct	143,000 00	Expenses	118,812.50
Income from Mdse .. .	237,625 00	Bal. Plant Acct .	474,500.00
Profit on Exchange	12 50	Bal. Current Acct	142,950.00
	$855,137 50		$855,137.50

PROFIT AND LOSS ACCOUNT

Expenses of Branch......	$118,812 50	Income of Branch ..	$237,625.00
Expenses of Home Office.	50,125 00	Profit on Exchange of	
Profit to Surplus.........	68,700 00	Branch....	12 50
	$237,637 50		$237,637 50

NEW YORK BALANCE SHEET

Assets

Current:				
Cash:				
New York	$100,000			
London	23,825	$123,825		
Customers—London.....		238,250		
Inventory—London................		119,125	$481,200	
Fixed:				
Plant—London			474,500	$955,700

Liabilities

Current:			
Creditors—London..................			238,250
			$717,450
Capital:			
Capital Stock		$500,000	
Surplus	$148,750		
Net Profits.	68,700	217,450	
			$717,450

A balance sheet of the London Branch is next set up, but this balance sheet needs no explanation.

In the above solution, note particularly the schedule of the current account and the method of finding the rate.

Various methods of taking up the profits of the branch on the Head Office books are used. Probably no explanation need be made, except as to the credit to the Profit and Loss account of the $12.50 profit on exchange. This might have been credited to an account called "Reserve for Fluctuation of Exchange."

Problems

1.* On December 31, the trial balance on the books of the London Office of the A. Rubber Company is as follows:

	£ s. d.	£ s. d.
Estate Purchase	3,000 0 0	
Estate Development	8,000 0 0	
Estate Produce Stock, Mar. 1.. . .	600 0 0	
Cash at Bank, London	800 0 0	
Estate Manager, Jan. 1	928 12 8	
Remittance to Estate Manager . . .	1,000 0 0	
London Office Expenses	400 0 0	
Share Capital..........		12,000 0 0
Creditors		1,900 0 0
Profit and Loss Balance...........		828 12 8
	£14,728 12 8	£14,728 12 8

* C. P. A. Examination.

After the above balances have been taken out, the accounts to December 31 are received from the estate manager, as follows (the dollar to be taken at 4s. 4d.):

Balance, Jan. 1	$ 4,286	
Remittances from London	8,600	
Rebates	131	
Sale of Produce	2,000	
Profit on Rice	249	
Expenditures on Development		$ 9,000
Expenditures on Purchase of New Land		2,800
Expenditures on Upkeep of Estate		1,646
Balance Carried Forward		1,820
	$15,266	$15,266

The produce unsold at December 31 was valued by the manager at $5,500.

You are required to construct the Revenue account and the balance sheet for presentation to the shareholders.

2. Change into dollars and cents the following items of a London Branch, and show the new value of the Head Office account:

Fixed assets	£6,000	Rates of exchange:	
Inventory (new)	500	Current	$4 $78\frac{1}{2}$
Cash	1,000	Average remittance	4 $62\frac{3}{16}$
Profit of period	1,500	Opening rate	4 $86\frac{3}{16}$
Head Office account	5,000	Average rate of year	4 $60\frac{7}{16}$
Remittance from Head Office	1,000	Balance on Head Office books	$27,200

PART II

CHAPTER 30

Compound Interest

Compound interest. In computing simple interest (Chapter 7), it has been seen that the principal remains constant. In computing compound interest, the principal is increased by the additions of interest at stated intervals. The total amount accumulated at the end of some given time is the compound amount, and the difference between the compound amount and the original principal is the compound interest. The principal of compound interest is logical, for if the periodic interest were paid to the lender, he would have this additional principal available for investment during the following period, and so on to the end of the last period.

Compound interest method. The compound interest method is the most accurate and scientific means of finding the true value of an investment. For this reason it is essential that accountants, and also investors in general, be familiar with it. The method is based on the foregoing assumption that all accumulations of interest become a part of the investment at the end of each interest period.

Actuarial science. Actuarial science is the mathematical science based upon compound interest and upon insurance probabilities (see Chapter 38). It deals with the investment of funds, and also with the mortality tables used by insurance companies (see Table 7, page 536, in the Appendix). The actuary uses tables for the greater part of his work: yet he must also have a knowledge of the fundamentals of his science. The accountant's interest in actuarial science is to give the best service to his clients by being able to compute investment values, prepare schedules of amortization, set up sinking fund accounts, and so forth.

Symbols. In the choice of the symbols used in this text, the attempt has been made to select those which are most commonly accepted.

The following symbols are given at this time for reference:

1 A unit of value, as $1, or the basis of any unit of value.

i The rate of interest for a single period.

j The nominal annual rate, if interest is compounded more often than once each year.

n The number of periods.
r The periodic ratio of increase, or $(1 + i)$.
m Frequency of periods during year.
s The compound amount of 1.
v The present worth of 1.
D The compound discount on 1 or the quantity of discount.
I The compound interest on 1 or the quantity of interest.
a_n The present value of an annuity of 1. (A is also used.)
R Rent, or periodic payment of an annuity.
s_n Amount of an annuity of 1.
P The principal.
S Any amount.

Principal. The principal is a sum of money or element of value for which interest is paid, or on which interest computations are based; 1 or \$1 will be used in most of the calculations in this work.

Time. The time which an investment has to run is commonly stated in years and months, but in compound interest computations it is better to state the time as a number of periods of equal duration. Thus, if a problem calls for "four years and six months, interest compounded semiannually," the time should be stated as nine periods of six months each.

Example

Time	*Frequency of Compounding*	*Number of Periods, or Value of* n
1 year	Annually	1
1 year	Semiannually	2
1 year	Quarterly	4
1 year	Monthly	12
3 years, 6 months	Semiannually	7
3 years, 6 months	Quarterly	14
5 years	Annually	5
5 years	Monthly	60

Rate. The rate is the measure of interest on the investment or principal. It may be indicated in different ways; for example, as .06, 6%, or $\frac{6}{100}$. The rate is generally stated as so much a year, but when the period of compounding is of any length other than a year, it is necessary to restate the rate as so much per period.

If the interest period is a half-year, it is necessary to divide the stated annual rate by two, and multiply the time in years by two; if the interest is compounded quarterly, it is necessary to divide the annual rate by four, and multiply the time in years by four.

Example

Time	*Rate*	*Frequency of Compounding*	*Number of Periods, or Value of* n	*Rate, or Value of* i
1 year	6%	Annually	1	6%
1 year	6%	Semiannually	2	3%
1 year	6%	Quarterly	4	$1\frac{1}{2}$%
1 year	6%	Monthly	12	$\frac{1}{2}$%
2 years	6%	Semiannually	4	3%
2 years	6%	Monthly	24	$\frac{1}{2}$%
3 years, 6 months	8%	Semiannually	7	4%
3 years, 6 months	4%	Semiannually	7	2%
3 years, 6 months	4%	Quarterly	14	1%

Ratio of increase. From every investment, the investor expects to receive the amount of his investment plus interest. He buys bonds, stocks, or other investments with the expectation of receiving an increased amount in return. If the total investment be multiplied by 1 plus the interest rate expressed decimally—as, for example, 1.06—the result will be the amount of his investment at the end of one interest period. The 1 plus the interest rate is called the ratio of increase.

Again, if the principal, \$1, is placed at interest at 6% for 1 year, it will be worth \$1.06 at the end of the year. The amount to be added to the \$1 is \$0.06. The ratio of increase is expressed as $(1 + .06)$, or 1.06; if the symbols previously given were used, the symbol would be $(1 + i)$, or r.

Compound amount tables. A compound amount table is a compilation of the value of 1 for various numbers of periods at various rates per cent. It is constructed by making the successive multiplications $(1.06)^1$, $(1.06)^2$, $(1.06)^3$, $(1.06)^4$, and so forth.

A compound amount table is given in Table 2, Appendix III, page 512.

When a compound amount table is available, many calculations may be eliminated. However, if one is not available, or if the factors to be used are not given in the table, the desired amount may be obtained by one of the methods described below.

Calculation of compound amount. The following methods of calculation are given to show the different means of arriving at the value of \$1 for a given time.

First method. The first method shows each step taken to find the value at the end of each period.

Example

Find the value of \$1 at 6% compound interest for 8 years.

Solution

1.00	$\times$ 1.06	=	1.06	for 1 period
1.06	$\times$ 1.06	=	1.1236	for 2 periods
1.1236	$\times$ 1.06	=	1.191016	for 3 periods
1.191016	$\times$ 1.06	=	1.262477	for 4 periods
1.262477	$\times$ 1.06	=	1.338225	for 5 periods
1.338225	$\times$ 1.06	=	1.418519	for 6 periods
1.418519	$\times$ 1.06	=	1.503630	for 7 periods
1.503630	$\times$ 1.06	=	1.593848	for 8 periods

It can be seen that the compound amount, 1.593848, is the sum of the investment and the compound interest.

The above method of arriving at the compound amount becomes very laborious when there are many periods.

Second method. In this method, multiplication is performed by using powers of numbers.

Example

Find the value of $(1 + i)^{12}$ or $(1.06)^{12}$.

Solution

Algebraic Increase

$$(1+i) \times (1+i) = (1+i)^2$$
$$(1+i)^2 \times (1+i)^2 = (1+i)^4$$
$$(1+i)^4 \times (1+i)^4 = (1+i)^8$$
$$(1+i)^8 \times (1+i)^4 = (1+i)^{12}$$

Arithmetical Increase

Exponents Added		*Powers of Ratio of Increase Multiplied*	
(1.06)	=	1.06	for 1 period
(1.06)	=	1.06	
$(1.06)^2$	=	1.1236	for 2 periods
$(1.06)^2$	=	1.1236	
$(1.06)^4$	=	1.262477	for 4 periods
$(1.06)^4$	=	1.262477	
$(1.06)^8$	=	1.593848	for 8 periods
$(1.06)^4$	=	1.262477	
$(1.06)^{12}$	=	2.012196	for 12 periods

The above principle may be applied for any power of a number. If the compound interest table does not extend to a sufficient number of periods, the required value may be found by this method, as illustrated in the following example:

Example

Find the compound amount of \$1 at 6% for 80 years.

Solution

Assume that we referred to the compound interest table, and found that the highest value shown at 6% was for 20 years, and was 3.2071355. The calculation for 80 years could be made as follows:

Exponents Added	*Powers of Ratio of Increase Multiplied*
$(1.06)^{20} =$	3.2071355
$(1.06)^{20} =$	3.2071355
$(1.06)^{40} =$	10.285718
$(1.06)^{40} =$	10.285718
$(1.06)^{80} =$	105.795994

Third method. The calculation by the third method is made by the use of logarithms.

Example

Find the value of $(1.06)^{80}$.

Solution

log 1.06	0 0253059
Multiply by exponent	80
log of 80th power of 1.06	2 0244720
Antilog 2.024472 .	105 80

Problems

Work the following problems by the methods indicated, and check your results by referring to the compound interest table. Show the complete work, as in the foregoing solutions.

Find the compound amount of:

1. $(1.04)^4$ by Method 1.
2. $(1.03)^6$ by Method 1.
3. $(1.06)^8$ by Method 1.
4. $(1.02)^6$ by Method 2.
5. $(1.005)^{30}$ by Method 2.
6. $(1.03)^{20}$ by Method 2.
7. $(1.06)^{24}$ by Method 2.
8. $(1.02)^{40}$ by Method 3.
9. $(1.005)^{50}$ by Method 3.
10. $(1.04)^{75}$ by Method 3.
11. $(1.04)^{60}$ by Method 3.
12. $(1.05)^{80}$ by Method 3.

Compound amount of given principal. To compute the compound amount of any principal, apply the following procedure.

Procedure: (*a*) Compute the compound amount of 1 for the number of periods at the given rate, $(1 + i)^n$, or s.

(*b*) Multiply the compound amount of 1 by the number of dollars in the investment, $P(1 + i)^n = S$.

Example

What will be the compound amount of $100 placed at interest at 6% for 4 years, interest compounded annually?

Formula	*Arithmetical Substitution*
$P(1 + i)^n = S$	$100(1.06)^4 = \$126.25$

Extended Solution

$1.00 \times 1.06 = 1.06$; or, amount of 1 for 1 year $= (1.06)^1$
$1.06 \times 1.06 = 1.1236$; or, amount of 1 for 2 years $= (1.06)^2$
$1.1236 \times 1.06 = 1.1910$; or, amount of 1 for 3 years $= (1.06)^3$
$1.1910 \times 1.06 = 1.2625$; or, amount of 1 for 4 years $= (1.06)^4$
$\$100 \times 1.2625 = \$126.25$; or, compound amount of \$100 for 4 years

Hereafter, instead of the compound amount of 1 at the end of each year being found as above, the solutions will be shortened by the use of the value of the expression $(1 + i)^n$, or s, as in the following. Find the value of s^4 at 6%.

Contracted Solution

$(1.06)^4 = 1.2625$, compound amount of 1 for 4 years
$\$100 \times 1.2625 = \$126.25$, compound amount of \$100 for 4 years

Use the compound interest table given in Table 2, Appendix III, page 512.

Problems

Construct formulas and write contracted solutions for the following:

	Principal	*Rate*	*Compounded*	*Years*
1.	\$ 600.00	6%	Annually	4
2.	\$ 400.00	3%	Annually	5
3.	\$ 600.00	5%	Annually	3
4.	\$1,000 00	4%	Annually	12
5.	\$ 256 25	6%	Annually	10
6.	\$1,247.50	3%	Annually	6
7.	\$3,847.50	4%	Annually	8
8.	\$1,472 25	3½%	Annually	4
9.	\$2,442.50	7%	Annually	10
10.	\$8,247.50	9%	Annually	20

Compound interest. Since an investment placed at interest for a definite time at a fixed rate will produce a given amount, the difference between this amount and the original investment will be the increase, or compound interest. To compute the compound interest, it is therefore necessary to find the amount at the end of the time and to deduct the principal from it.

Procedure: (*a*) Determine the compound amount of 1 for the number of periods at the given rate, $(1 + i)^n = s$.

(*b*) Find the compound interest on 1 by deducting 1 from the compound amount of 1, $s - 1 = I$.

(*c*) Multiply the compound interest on 1 by the number of dollars in the investment, $P(s - 1) = I$.

Example

Find the compound interest on \$100.00 for 4 years at 6%.

Formula	*Arithmetical Substitution*
$P(s - 1) = I$	$100(1.2625 - 1) = \$26.25$

Solution

$(1.06)^4 = 1.2625$, compound amount of 1 for 4 years
$1.2625 - 1 = .2625$, compound interest on 1 for 4 years
$\$100 \times .2625 = \$26.25$, compound interest on \$100 for 4 years

TABLE OF ANALYSIS OF COMPOUND INTEREST

(1) *End of Period*	(2) *Principal*	(3) *Simple Interest*	(4) *Interest on Interest*	(5) *Compound Interest*	(6) *Compound Amount*
1	\$1 00	\$ 06	\$...........	\$ 06	\$1 06
2	1 00	.12	.0036	.1236	1 1236
3	1 00	.18	.011016	.191016	1 191016
4	1.00	.24	.022477	.262477	1.262477

Problems

1. Find the compound interest on \$500 for 5 years at 3%.
2. Find the compound interest on \$650 for 6 years at 4%.
3. Find the compound interest on \$2,560 for 4 years at 6%.
4. Construct a table of analysis of the compound interest on \$800 at 4% for 4 years.
5. Construct a table showing the complete analysis of \$447.20 at 5% compound interest for 4 years.

Results of frequent conversions of interest. Compound interest is usually stated as a certain rate per annum, but if the interest is to be compounded more often than once each year, the total accumulation will be greater than the accumulation of $(1 + i)^n$ times the principal, where i is used as the annual rate and n as the number of years.

A study of the following will show the difference between the amount of \$100 at 6% interest, compounded monthly for 10 years, and the amount of \$100 at 6% interest compounded annually for 10 years.

$100 \times (1.005)^{120}$ (compounded monthly for 10 years)....	\$181 94
$100 \times (1.06)^{10}$ (compounded annually for 10 years)...	179.08
Difference caused by frequent conversions	\$ 2 86

Nominal and effective rates. Nominal, as the word implies, is defined as "in name only." In illustration (*a*), above, 6% is the nominal rate. Effective interest is the interest actually received by the investor, and is based upon the amount invested and upon 1 year as the period of time. In illustration (*a*), \$6.17 is the amount of interest received in 1 year on an investment of \$100; in effect, this is 6.17% on the investment, or an effective rate of 6.17%.

In order to distinguish between nominal and effective rates, the following symbols are used:

i = the effective annual rate
j = the nominal annual rate
m = the number of conversions each year

The procedure in calculating the effective rate, the nominal rate being given, is as follows:

Procedure: (*a*) Find the number of conversion periods each year; (*b*) find the rate per period; (*c*) determine the compound amount of 1 for the number of periods found in (*a*) and at the rate per period found in (*b*); (*d*) deduct 1 from the compound amount of 1.

Example

What is the annual effective rate if the nominal rate is 6%, compounded quarterly?

Formula	*Arithmetical Substitution*
$\left(1 + \frac{j}{m}\right)^{mn} - 1 = i$	$\left(1 + \frac{.06}{4}\right)^{4} - 1 = .061364$

Solution

$1 \times 4 = 4$, number of periods
$.06 \div 4 = .015$, rate per period
$1 \times 1.015 = 1.015$, compound amount of 1 for 1 period at 1.5%
$1.015 \times 1.015 = 1.030225$, compound amount of 1 for 2 periods at 1.5%
$1.030225 \times 1.030225 = 1.061364$, compound amount of 1 for 4 periods at 1.5%
$1.061364 - 1 = .061364$, or 6.1364%, annual effective rate

The procedure in calculating the nominal rate, the effective rate being given, is as follows.

Procedure: (*a*) Determine the log of 1 plus the effective rate, or the ratio of increase.

(*b*) Divide the log found in (*a*) by the number of periods of compounding.

(*c*) Find the antilog of the quotient of (*b*), to determine 1 plus the nominal rate.

(*d*) Deduct 1 to determine the nominal rate.

Example

An insurance company receives 6% effective interest on a certain investment· what is the nominal rate per annum, if interest is compounded quarterly?

Formula	*Arithmetical Substitution*
$[\sqrt[n]{(1 + i)} - 1]mn$ = Nominal rate	$[\sqrt[4]{(1.06)} - 1]4 = .058788$

Solution

$$
\begin{aligned}
\log (1.06) &= 0.0253059 \\
0.0253059 \div 4 &= 0.0063265, \text{ log of ratio of increase} \\
\text{antilog } 0.0063265 &= 1.014672, \text{ ratio of increase} \\
1.014672 - 1 &= .014672, \text{ rate of increase for 1 quarter} \\
.014672 \times 4 &= .058788, \text{ or } 5.8788\%, \text{ nominal rate per annum}
\end{aligned}
$$

NOTE. $\sqrt[4]{(1.06)}$ is sometimes written $(1.06)^{\frac{1}{4}}$.

Effective interest. While calculations of the effective rate and of the nominal rate are always based on a unit period of 1 year, most investments are for periods of more than 1 year. The added feature of a number of years may be included in the calculation by either of two methods:

(1) Find the effective rate for the year, and use the actual number of years for the period of investment.

(2) Find the rate for one period, and also the number of periods, and use these results as the values of i and n. See page 312 under "rate."

The use of the first, or effective interest method, is advantageous in some annuity computations. However, because of its simplicity the second method is the one most commonly used; it will be employed hereafter unless it is necessary to use the first method for explanatory purposes.

First method.

Procedure: (*a*) Calculate the effective rate per year.

(*b*) Find the compound amount for the number of years.

(*c*) Multiply the compound amount by the principal in dollars.

Example

What amount will be due in 5 years if $200 is placed at interest at 5%, compounded quarterly?

Solution

$$\left(1 + \frac{.05}{4}\right)^{5} - 1 = .050945, \text{ effective rate for 1 year}$$

$$
\begin{aligned}
(1.050945)^5 &= 1.282037, \text{ compound amount for 5 years} \\
1.282037 \times \$200 &= \$256.41, \text{ compound amount of \$200 for 5 years}
\end{aligned}
$$

Second method.

Procedure: (*a*) Find the total number of periods.

(*b*) Find the rate per period.

(*c*) Determine the compound amount of 1 for the number of periods at the rate found in (*a*).

This method is preferable when interest tables are available.

Example

What amount will be due in 5 years if \$200 is placed at interest at 5%, compounded quarterly?

Solution

The number of interest periods is 4×5, or 20. The rate of interest per period is $.05 \div 4$, or .0125.

$(1.0125)^{20} = 1.282037$, compound amount for 20 periods at 1.25%

$1.282037 \times \$200 = \$256.41$, compound amount of \$200 for 5 years at 5%, compounded quarterly

Problems

1. Find the compound amounts of the following:
   (*a*) \$1,500 at 2% compounded quarterly for 5 years.
   (*b*) \$450.25 at 4% compounded semiannually for 8 years.
   (*c*) \$1,250 at 3% compounded quarterly for 10 years.

2. Find the effective rate equivalent to:
   (*a*) 4% compounded quarterly.
   (*b*) 7% compounded semiannually.
   (*c*) 8% compounded quarterly.
   (*d*) 6% compounded monthly.

3. Construct a table, similar to the one on page 317, for 4 years at 3%, interest to be compounded semiannually.

Compound present worth. Sometimes it is desired to find what principal placed at interest now will amount to a certain sum at a definite future time.

The present worth of a sum which is due at the end of a certain number of periods is a smaller sum which, if put at compound interest at a given rate, will amount to the known sum in the given time.

The ratio of increase employed in accumulating 1 to a compound amount is the same as the ratio of increase employed in accumulating a present worth to 1. To illustrate:

Compound Amount		*Present Worth*	
Basis of calculation:	\$1.00	Present worth of \$1:	\$ .792094
Multiplying:	1.06	Multiplying:	1 06
	\$1.06		\$ .839619
Multiplying:	1.06	Multiplying:	1 06
	\$1.1236		\$ .889996
Multiplying:	1.06	Multiplying:	1.06
	\$1.191016		\$ .943396
Multiplying:	1.06	Multiplying:	1.06
Compound amount:	\$1.262477	Basis of calculation:	\$1.000000

$1 \div 1.262477 = .792094$, present worth

or

$1 \div .792094 = 1.262477$, compound amount

Procedure: (*a*) Compute the present worth by dividing 1 by the compound amount of 1, $1 \div s = v^n$.

(*b*) Multiply the present worth of 1 by the number of dollars to be produced, $S \times v^n = P$.

Example

What amount of money, invested at compound interest at 6% for 4 years, will produce $100?

Formula	*Arithmetical Substitution*
$S \times v^n = P$	$100 \times .7921 = \$79.21$

Solution

$(1.06)^4 = 1.262477$, compound amount of 1 for 4 years at 6%
$1 \div 1.262477 = .7921$, compound present worth of 1 for 4 years at 6%
$\$100 \times .7921 = \$79.21$, compound present worth of $100 for 4 years at 6%

Verification

$79.21 \times 1.06 = \$\ 83.96$, compound amount for 1 year
$83.96 \times 1.06 = \$\ 89.00$, compound amount for 2 years
$89.00 \times 1.06 = \$\ 94.34$, compound amount for 3 years
$94.34 \times 1.06 = \$100.00$, compound amount for 4 years

TABLE OF COMPOUND PRESENT WORTH OF 1

(1) *End of Period*	(2) *Principal*		(3) *Compound Amount (Inverted Order)*		(4) *Present Worth of 1*
	$1.00	÷	$1.262477	=	$ .792094
1	1 00	÷	1.191016	=	.839619
2	1.00	÷	1.1236	=	.889996
3	1.00	÷	1.06	=	943396
4	1.00	÷	1 00	=	1.000000

Problems

Set up formulas, solutions, and verifications for the following:

1. What amount placed in the bank at 4%, interest compounded semiannually, will accumulate to $2,000 in 5 years?

2. What principal will have to be placed at interest at $3\frac{1}{2}$%, compounded semiannually, to accumulate to $3,000 in 3 years?

3. What amount of money will have to be placed on deposit to cancel a debt of $2,375.50 due in 5 years without interest, if the amount deposited is to be credited with interest at 4%, compounded quarterly?

4. Construct a table, similar to the one above, for $1 at 4% for 5 years, interest compounded semiannually.

Compound discount. The compound discount is the difference between 1 and the present worth of 1.

Procedure: (*a*) Calculate the compound discount by deducting from 1 the present worth of 1, $(1 - v^n) = D$.

(*b*) Multiply the compound discount on 1 by the number of dollars, $S(1 - v^n) = D$.

Example

What is the compound discount on \$100 due in 4 years, if money can be invested at 6%, interest compounded annually?

Formula	*Arithmetical Substitution*
$S(1 - v^n) = D$	$100(1 - .792094) = \$20.79$

Solution

$(1.06)^4 = 1.262477$, compound amount of 1 for 4 years at 6%
$1 \div 1.262477 = .792094$, compound present worth of 1 due at the end of 4 years at 6%
$1 - .792094 = .207906$, compound discount on 1 due at the end of 4 years at 6%
$\$100 \times .207906 = \$20.79$, compound discount on \$100 due at the end of 4 years at 6%

Problems

Construct formulas and write solutions for the following:

1. Find the compound discount on \$500 due in 4 years, money being worth 5%.

2. Compute the compound discount on \$600 due in 5 years, money being worth $4\frac{1}{2}$%.

3. Required, the compound discount on \$800 due in 10 years, money being worth 4%.

Rate. The rate of interest may be computed if the principal, the amount, and the time are known. The computation involves the use of a principle illustrated in the chapter on logarithms (see page 256).

Procedure: (*a*) Calculate the compound amount of 1 by dividing the given compound amount by the principal.

(*b*) Determine the log of the compound amount of 1.

(*c*) Divide the log of the compound amount of 1 by the number of periods.

(*d*) Determine the ratio of increase by finding the antilog of (*c*).

(*e*) Deduct 1 from the ratio of increase.

Example

If \$100 amounts to \$126 25 in 4 years, what is the rate of interest?

Formula	*Arithmetical Substitution*
$\sqrt[n]{\frac{\text{Amount}}{\text{Principal}}} - 1 = i$	$\sqrt[4]{\frac{126.25}{100.00}} - 1 = .06$

Solution

$$
\begin{aligned}
126.25 \div 100 &= 1.2625\text{, compound amount of 1 for 4 years} \\
&\quad\ \text{at the unknown rate} \\
\log 1.2625 &= 0.101231 \\
0.101231 \div 4 &= 0.025307 \\
\text{antilog of } 0.025307 &= 1.06\text{, ratio of increase} \\
1.06 - 1 &= .06\text{, or } 6\%\text{, the rate}
\end{aligned}
$$

Verification

$$
\begin{aligned}
1.00 \times 1.06 &= 1.06 \\
1.06 \times 1.06 &= 1.1236 \\
1.1236 \times 1.06 &= 1.1910 \\
1.1910 \times 1.06 &= 1.2625 \\
\$100.00 \times 1.2625 &= \$126.25
\end{aligned}
$$

Problems

1. If \$100 amounts to \$130.70 in 5 years, what is the rate of interest?

2. If \$1,000 amounts to \$1,127.16 in 2 years, what is the rate of interest, compounded monthly? Set up the formula, the solution, and the verification.

3. Compute the annual rate of interest for each of the following:

	Principal	*Amount*	*Time*
(*a*)	\$ 100	\$ 133.82	5 years
(*b*)	200	310 59	10 years
(*c*)	80	212.26	20 years
(*d*)	1,000	2,830.75	25 years
(*e*)	40	68.10	18 years

4. At what nominal rate of interest per annum will \$200 amount to \$268.78 in 5 years, if interest is converted semiannually?

5. At what rate of interest will any principal double itself in 10 years?

Time. By applying the principles of logarithms, the time may be computed if the principal, the amount, and the rate are given.

Procedure: (*a*) Determine the compound amount of 1 by dividing the compound amount by the principal.

(*b*) Determine the log of the compound amount of 1.

(*c*) Determine the log of the ratio of increase.

(*d*) Divide (*b*) by (*c*), to determine the time in periods.

Example

If \$100 placed at interest at 6%, compounded annually, amounts to \$126.25, what is the time of the investment?

Formula

$$\frac{\log \text{ of } \dfrac{\text{Amount}}{\text{Principal}}}{\log \text{ of } (1 + i)} = n$$

Arithmetical Substitution

$$\frac{\log \text{ of } \dfrac{126.25}{100.00}}{\log \text{ of } (1.06)} = 4n$$

Solution

$$
\begin{aligned}
126.25 \div 100 &= 1.2625, \text{ compound amount of 1 at } 6\% \text{ for } n \text{ periods}\\
\log 1.2625 &= 0.101231\\
\log 1.06 &= 0.025306\\
0.101231 \div 0.025306 &= 4, \text{ time in periods, or 4 years}
\end{aligned}
$$

Problems

Compute the time in each of the following:

	Principal	*Amount*	*Rate*	*Convertible*
(*a*)	$ 100	$ 240.66	5%	Annually
(*b*)	1,000	3,207.14	6%	"
(*c*)	200	533.17	4%	"
(*d*)	40	62.32	3%	"
(*e*)	500	1,621.70	4%	"
(*f*)	300	609.84	6%	Semiannually
(*g*)	100	200.00	5%	"

Compound amount for fractional part of conversion period. In the problems thus far, the time contained an exact number of conversion intervals. How shall compound interest be computed when there is a fractional part of an interest period, for example, if the time is 4 years, 2 months, interest at 6% convertible semiannually?

In actual practice, simple interest is customarily used for fractions of an interest period. In the example above: (*a*) compound interest would be computed for 8 years at 3%; (*b*) simple interest would be computed on the amount found in (*a*) at 3% for 2 months; and (*c*) the sum of the answers found in (*a*) and (*b*) would be the amount due.

Problems

1. Find the compound amount of $250 for 3 years and 3 months, interest at 5% converted annually.

2. Find the compound amount of $1,575 for 4 years and 3 months, interest at 5% converted semiannually.

3. Find the present value of $2,750 due in 2 years, 8 months, if the interest rate is 6%, compounded semiannually.

Review Problems

1. How much money will have accumulated after 8 years if $500 is invested now at 4% converted quarterly?

2. Calculate the value of v^5 at 3% by dividing 1 by $(1.03)^5$ as shown by the compound amount table. Compute the value of v^5 at 3% by logarithms.

3. If the annual interest rate is 5%, what is the corresponding rate of discount? $\left(\text{Hint: } d = \dfrac{i}{1+i}.\right)$

4. How long will it take money to double itself when invested at 3% converted annually?

5. What nominal rate, convertible monthly, is equivalent to 4% a year effective?

6. Find the present value of a debt of \$750 due in 5 years, if the current rate of interest is 6% convertible monthly.

7. If you can get 4% converted annually, how much will you need to invest to accumulate \$7,500 in 15 years?

8. Find the present value of \$1,250 due in 4 years, 6 months, with interest at 4% convertible semiannually.

9. At age 42, a man has \$6,725 to invest. What interest rate, converted annually, must he receive in order for the investment to accumulate to \$12,500 at age 60?

10. Calculate by means of logarithms the present value of \$382.45 due in 5 years without interest, if money is worth 4% effective.

CHAPTER 31

Ordinary Annuities

Definition. An annuity is a series of equal payments made at equal intervals of time. Examples of annuities are: premiums on life insurance, interest payments on bonds or mortgages, rentals of property, pensions, sinking fund payments, regular preferred stock dividends, and so forth.

The word *annuity* suggests annual payments, but the broad meaning of the term is a series of equal payments made at equal stated intervals, whether these intervals are one year, six months, three months, or any other period of time.

Kinds of annuities. Annuities are of two kinds:

(1) *Ordinary annuity.* This is a series of payments where each periodical payment is made at the end of a period.

(2) *Annuity due.* This is a series of payments where each periodical payment is made at the begining of the period.

Ordinary annuities will be discussed in this chapter and annuities due in the next chapter.

Rent of an annuity. The periodic payments are known as "rents," and the single periodic payment is represented by the symbol R.

Amount of an ordinary annuity. The amount of an annuity of 1 for any number of periods (n) is represented by s_n (read "s sub n"). This symbol used in conjuction with the rate of interest becomes $s_{\overline{n}|i}$; for example, $s_{\overline{10}|5\%}$ represents the amount of an annuity of 1 a period, for 10 periods, at 5%. The amount of an ordinary annuity is the sum at the end of the term of all the periodic payments, plus the interest on all payments.

Analysis of compound interest. To understand annuities, it is necessary to know how compound interest tables are built up. Take $(1 + i)^n = (1.06)^4$. This amount can be found in the 6% column of a compound interest table, the fourth number from the top. An analysis shows that it is composed of three elements: a principal of 1; an annual addition of .06 simple interest; and "interest on interest" of .02247696. A further analysis may be made as shown on the next page.

	Initial Payment	*End of 1st Period*	*End of 2nd Period*	*End of 3rd Period*	*End of 4th Period*	*Total*
Invested.........	$1.00					$1.00
1st annual interest on 1...........		.06				
Interest on .06....			.0036	.0036	0036	
Interest on 1st .0036				.000216	000216	
Interest on 2nd .0036..					.000216	
Interest on .000216					.00001296	.07146096
2nd annual interest on 1..			.06			
Interest on 06 . .				.0036	0036	
Interest on .0036 .					.000216	.067416
3rd annual interest on 1...........				.06		
Interest on .06....					.0036	.0636
4th annual interest on 1...........					.06	.06
Total........						1 26247696

When the above tabulation is summarized, three distinct parts appear:

1. The principal...	$1.00
2. The four equal annual amounts of simple interest, $.06.....	.24
3. The accumulations of "interest on interest".....	.02247696
Total..................................	$1 26247696

This may be further reduced to:

1. Principal	$1 00
2. Compound interest.........................	26247696
3. Compound amount.........................	$1.26247696

Relation of compound interest and annuities. In the above table of analysis, it can be seen that there is a series of payments of $.06 each at regular stated intervals of 1 year, and that compound interest is calculated on each of these $.06 payments until the end of the fourth year. Therefore, .26247696 is the amount of an annuity of .06 for 4 years at 6%.

An annuity of 1 may be computed from this result as follows:

.26247696 ÷ 6 = .04374616, amount of an annuity of .01 for 4 years at 6%

.04374616 × 100 = 4.374616, amount of an annuity of 1 for 4 years at 6%

Procedure in computing the amount of an annuity. From the foregoing discussion may be derived the following general pro-

cedure for computing the amount of an annuity for any given number of periods at any stated interest rate:

Procedure: (*a*) Calculate the compound amount at the periodic rate and for the number of periods given, or obtain the compound amount from a compound amount table, $(1 + i)^n = s$.

(*b*) Deduct 1 from the compound amount, $s - 1 = I$.

(*c*) Divide the compound interest on 1 by the rate per cent expressed decimally, to obtain the amount of the annuity of 1, $I \div i = s_{\overline{n}|i}$.

(*d*) Multiply the amount of the annuity of 1 by the number of dollars of each annuity rent, $R \times s_{\overline{n}|i} = S$.

Example

It is desired to find the amount of an ordinary annuity of \$100 for 5 years at 6%.

Solution

$$
\begin{aligned}
1.338226 &= \text{compound amount of 1 at 6\%, or } (1.06)^5 \\
1.338226 - 1 &= .338226\text{, compound interest on 1 at 6\%} \\
.338226 \div .06 &= 5.6371\text{, amount of annuity of 1} \\
\$100 \times 5.6371 &= \$563.71\text{, amount of annuity of \$100 for 5 years} \\
&\quad \text{at 6\%}
\end{aligned}
$$

From the above, the following may be derived:

$$
\begin{aligned}
(1 + i)^n\text{, or } (1.06)^5 &= 1.338226\text{, compound amount, or } s. \\
(1 + i)^n - 1\text{, or } (1.06)^5 - 1 &= .338226\text{, compound interest, or } I. \\
\frac{(1 + i)^n - 1}{i}\text{, or } \frac{(1.06)^5 - 1}{.06} &= 5.6371\text{, amount of an annuity of 1, or } s_{\overline{n}|i} \\
Rs_{\overline{n}|i} = S \quad \text{or} &\quad \$100 \times 5.6371 = \$563.71
\end{aligned}
$$

Verification

Rent paid at end of first year		\$100.00
Interest at 6% on \$100	\$ 6.00	
Rent paid at end of second year	100.00	106.00
Amount of annuity for 2 years		\$206.00
Interest at 6% on \$206	\$ 12.36	
Rent paid at end of third year	100.00	112 36
Amount of annuity for 3 years		\$318.36
Interest at 6% on \$318.36	\$ 19.10	
Rent paid at end of fourth year	100.00	119 10
Amount of annuity for 4 years		\$437.46
Interest at 6% on \$437.46	\$ 26 25	
Rent paid at end of fifth year	100.00	126.25
Total		\$563.71

Semiannual or quarterly basis. If the rents are payable every six months, or every three months, and the interest is to be compounded on the same dates, the method of using the rate per period

and the time in periods is preferable to the method of finding the effective interest for 1 year and using the time in years. Only the former method will be illustrated.

Procedure: (*a*) Find the nominal rate per period, $\frac{j}{m}$.

(*b*) Find the number of periods, mn.

(*c*) Determine the compound amount, $\left(1 + \frac{j}{m}\right)^{mn}$.

(*d*) Determine the compound interest on 1 for the number of periods at the nominal rate per period, $\left(1 + \frac{j}{m}\right)^{mn} - 1 = I$.

(*e*) Divide the compound interest found in (*d*) by the nominal rate per cent per period, $I \div \frac{j}{m} = s_{\overline{n}|i}$.

(*f*) Multiply the amount of the annuity of 1 by the number of dollars of each periodic rent, $Rs_{\overline{n}|i}$.

Example

What will be the amount of an ordinary annuity of 8 rents of $50 each, payable every 6 months, interest at 6% per year, compounded semiannually?

Formula	*Arithmetical Substitution*
$Rs_{\overline{n}\|i} = S$	$50\left(\frac{(1.03)^8 - 1}{.03}\right) = \$444.62$

Solution

$$
\begin{aligned}
4 \times 2 &= 8, \text{ number of periods} \\
.06 \div 2 &= .03, \text{ rate per cent per period} \\
(1.03)^8 &= 1.26677, \text{ 1 at compound interest for 8 periods} \\
1.26677 - 1 &= .26677, \text{ compound interest on 1 for 8 periods} \\
.26677 \div .03 &= 8.8923, \text{ amount of annuity of 1 for 8 periods, or } s_{\overline{8}|3\%}.^* \\
\$50 \times 8.8923 &= \$444.62, \text{ amount of annuity of \$50 for 8 periods}
\end{aligned}
$$

Problems

1. Find the amount of an ordinary annuity of:

(*a*)	$200	at 5%	for 4	years, rents and interest payable	annually.
(*b*)	$120	" 4%	" 6	" " " " "	"
(*c*)	$250	" 3%	" 20	" " " " "	semiannually.
(*d*)	$250	" 6%	" 10	" " " " "	"
(*e*)	$500	" 4%	" 30	" " " " "	annually.
(*f*)	$100	" 6%	" 40	" " " " "	"

2. Construct a table of analysis of the amount of an annuity of 1 for 5 years at 4%.

3. If for 5 years $250 is deposited at the end of every six months in a bank paying $3\frac{1}{2}$%, interest converted semiannually, what will be the amount credited to the account at the end of the term? Set up a schedule showing: (*a*) number

---

*The value of $s_{\overline{8}|3\%}$ may be found in Table 4, page 527.

of periods; (*b*) amount deposited; (*c*) interest each period; (*d*) amount to be added to the account; (*e*) balance of the account each period.

4. *A* purchases a house, and agrees to pay \$60 each month for 1 year. If money is worth 6%, interest compounded monthly, what sum paid in one amount at the end of the year would be the equivalent of *A*'s total monthly payments?

Rent of an ordinary annuity. Frequently the amount of an ordinary annuity is known and it is desired to find the periodic rent, as in problems of sinking funds.

Procedure: (*a*) Compute the amount of the annuity of 1 for the given number of periods at the given rate per period, $s_{\overline{n}|i}$.

(*b*) Divide the number of dollars of the required amount by the amount of an annuity of 1. The result will be the rent, or periodic payment, $P \div s_{\overline{n}|i} = R$.

Example

What should be the amount of each equal annual payment into a fund which, in 4 years at 6%, interest compounded annually, is to amount to \$1,000?

Solution

$$P \div s_{\overline{n}|i} = R \qquad 1000 \div s_{\overline{4}|6\%} = R$$

In Table 4 it will be found that

$s_{\overline{4}|6\%}$ = 4.374616, amount of annuity of 1 for 4 years at 6%

\$1,000 ÷ 4.374616 = \$228.59, rent required to accumulate \$1,000 at the end of 4 years

Verification

Rent at end of first year	\$ 228 59
Interest on \$228.59 for 1 year at 6%	13 72
Rent at end of second year	228 59
Amount of annuity at the end of 2 years	\$ 470 90
Interest on \$470.90 for 1 year at 6%	28.25
Rent at end of third year	228 59
Amount of annuity at the end of 3 years	\$ 727 74
Interest on \$727.74 for 1 year at 6%	43 66
Rent at end of fourth year	228 60
Amount of annuity at the end of 4 years at 6%	\$1,000 00

TABLE OF AMOUNT OF ANNUITY

(1) *End of Period*	(2) *Rent*	(3) *Interest Accumulation*	(4) *Addition to Principal*	(5) *Total Amount*
1	\$228 59	\$........	\$228 59	\$ 228 59
2	228 59	13 72	242 31	470.90
3	228 59	28.25	256.84	727.74
4	228.60	43.66	272.26	1,000.00

Problems

1. A savings bank pays 2%, interest compounded quarterly. How much must be deposited at the end of each quarter in order to accumulate \$400 at the end of 2 years? Prepare a table for verification.

2. A company owes \$600, due in 4 years. How much must be set aside semiannually at 4%, interest compounded semiannually, to accumulate to the amount of the debt at maturity? Prepare formula, solution, and table.

3. A company issued bonds for \$30,000, due in 10 years. Interest is at 5%, compounded quarterly. How much must the company set aside every three months in order to be able to meet the payments on the bonds when they become due?

4. A company has a debt of \$20,000, due at the end of 10 years. Money is worth 5%, interest compounded annually. How much must be set aside annually to accumulate to the amount of the debt?

5. At the age of 30, Y decides that he ought to deposit in the bank, every three months, an amount which will have accumulated to \$25,000 by the time he is 55. The bank allows him 4%, interest compounded quarterly. What is the amount of Y's quarterly deposits?

Use of effective interest in annuities. Very often the rents are paid annually, and the interest is compounded semiannually or quarterly; when such is the case, the effective rate of interest must be used.

If the interest is compounded more or less frequently than the rents are paid, it is necessary to convert the nominal interest rate to the effective interest rate applicable to the rent periods.

Procedure: (*a*) Calculate the effective periodic interest on the basis of the rent periods, $\left(1 + \frac{j}{m}\right)^m - 1$.

(*b*) Calculate the amount of an annuity of 1, using the effective rate per period; the number of periods corresponds to the number of rents, $s_{\overline{n}|i}$.

(*c*) Multiply the amount of an annuity of 1, found in (*b*), by the number of dollars of each rent, $Rs_{\overline{n}|i} = S$.

Example

What will be the amount of an annuity, the annual payments of which are \$100 for 4 years at 6%, interest compounded quarterly?

Solution

The solution to this example will be stated in two parts

(1) Calculation of the effective rate per year.
(2) Calculation of the amount of the annuity of \$100.

PART 1

Formula — *Arithmetical Substitution*

$$\left(1+\frac{j}{m}\right)^m - 1 = \text{Effective rate} \qquad \left(1+\frac{.06}{4}\right)^4 - 1 = .0613635$$

$.06 \div 4 = .015$, rate per period
$1 + .015 = 1.015$, ratio of increase
$(1.015)^4 = 1.0613635$, compound amount of 1 for 1 year
$1.0613635 - 1 = .0613635$, effective rate per year

PART 2

Use the effective rate found in Part 1, and proceed as in the examples previously given.

Formula — *Arithmetical Substitution*

$$Rs_{\overline{n|}i} = S. \qquad 100\left(\frac{(1.0613635)^4 - 1}{.0613635}\right) = \$438.35$$

$(1.0613635)^4 = 1.2689855$, compound amount of 1 for 4 periods at 6.13635%*
$1.2689855 - 1 = .2689855$, compound interest on 1 for 4 periods at 6.13635%
$.2689855 \div .0613635 = 4.383477$, amount of annuity of 1 for 4 periods at 6.13635%
$\$100 \times 4.383477 = \$438.35$, amount of annuity of \$100 for 4 years

Verification

First year:		
Rent at end of year		\$100 00
Second year:		
\$100 × .0613635 (effective rate)	\$ 6 14	
Rent	100 00	106 14
Amount of annuity for 2 years		\$206.14
Third year:		
\$206.14 × .0613635	\$ 12 65	
Rent	100 00	112 65
Amount of annuity for 3 years		\$318.79
Fourth year:		
\$318.79 × .0613635	\$ 19 56	
Rent	100 00	119 56
Amount of annuity for 4 years		\$438 35

Problems

1. Barlow has a 5-year annuity for which the payments, made at the end of each year, are \$300 each. Interest is at 4%, compounded semiannually. What is the amount of the annuity? Prepare proof of answer.

2. Ware desires to know how much he will have in the savings bank at the end of 25 years if he deposits \$150 at the end of each six months. The bank pays 4%, and the interest is compounded at the end of each quarterly period.

* The compound amount of 1 for 4 periods at 6.13635% is the same as the compound amount of 1 for 16 periods at $1\frac{1}{2}\%$, and $(1.015)^{16}$ is readily found in the compound amount table given in Appendix III.

3. Deposits of $500 are made at the end of each year for 20 years. If the bank credits the account with quarterly interest at 4% (nominal rate), what will be the amount of the accumulation?

Sinking fund contributions. A sinking fund produced by equal periodic payments accumulating at compound interest is one type of annuity. The principles applicable to annuities will be further illustrated with special reference to sinking funds.

To find the rent of a sinking fund, divide the number of dollars required in the total fund by the amount of the annuity of 1 for the specified number of periods at the given rate, $P \div s_{\overline{n}|i} = R$.

Example

A company borrowed $2,500 for 5 years, and established a sinking fund to provide for the payment of the debt. The contributions to the fund were to be made at the end of each year. If money is worth 6%, what should be the amount of each annual contribution?

Solution

$$P \div s_{\overline{n}|i} = R \qquad \$2{,}500 \div s_{\overline{5}|6\%} = \$443.49$$

In Table 4 it will be found that

$s_{\overline{5}|6\%} = 5.63709$, amount of ordinary annuity of 1 for 5 periods at 6%

$2,500 ÷ 5.63709 = $443.49, contribution to sinking fund.

TABLE OF ACCUMULATION OF SINKING FUND CONTRIBUTIONS

(1) *End of Period*	(2) *Contribution*	(3) *Interest*	(4) *Yearly Increase*	(5) *Total Fund*
1	$ 443 49	$...........	$ 443 49	$ 443 49
2	443 49	26 61	470 10	913 59
3	443 49	54 82	498 31	1,411 90
4	443 49	84 71	528 20	1,940 10
5	443 49	116 41	559 90	2,500 00
	$2,217.45	$282 55	$2,500 00	

Problems

1. A company establishes a sinking fund to provide for the payment of a debt of $8,000 maturing in 4 years. The contributions to the fund are to be made at the end of each six months. Interest at 4% is to be compounded semiannually. What must be the amount of each semiannual contribution? Construct a table, as in the example above.

2. A debt of $30,000 is due in 4 years. A sinking fund is to be established, and contributions are to be made at the end of each six months. What must be the amount of each semiannual contribution, if interest at 4% is compounded semiannually? Construct a table, as in the example above.

3. *A* has an obligation of $8,000 maturing in 3 years. How much must he set aside each month at 6%, interest compounded monthly, in order to be able to pay the debt when due?

Present value of an ordinary annuity. The present value of an ordinary annuity represented by the symbol $a_{\overline{n}|i}$ is the sum which, if put at compound interest, will produce the periodic rents of the annuity contract as they become due.

First method. Procedure: Find the present value of each periodic rent separately; add the present values of these periodic rents; their sum is the present value of the annuity.

Example

Assume that it is desired to find the value, at the beginning of the first period, of a series of four annual periodic payments of 6¢ each. Respective payments are to be made at the end of each year.

PRESENT VALUE AT DATE OF CONTRACT

1st rent, payable at end of 1st year ...	$.06 \times \frac{1}{(1.06)}$	$= .06 \times .94339 =$	056603
2nd rent, payable at end of 2nd year...	$.06 \times \frac{1}{(1.06)^2}$	$= .06 \times .88999 =$	.053399
3rd rent, payable at end of 3rd year...	$.06 \times \frac{1}{(1.06)^3}$	$= .06 \times .83962 =$	.050377
4th rent, payable at end of 4th year...	$.06 \times \frac{1}{(1.06)^4}$	$= .06 \times .79209 =$	.047525
Present value of an annuity of .06		=	207904

Second method. It will be noted that the present value at 6% of an annuity of 4 rents of 6¢ each is the same as the compound discount on 1 for 4 periods at 6%. The computation of the compound discount on 1 for 4 periods at the rate of 6% is as follows:

$(1.06)^4 = 1.262477$, compound amount of 1 for 4 periods at 6%
$1 \div 1.262477 = .792094$, present value of 1 for 4 periods at 6%
$1 - .792094 = .207904$, compound discount on 1 for 4 periods at 6%

Substituting symbols for figures, it is apparent that the present value of an annuity for n periods, the rents of which when stated in cents are the same as i, is equal to the compound discount on 1 for n periods at the rate of i.

In the example just given, the basis of calculation is the periodic payment of 6¢, and this is stated as "an annuity of 6¢." However, the basis most frequently used, or the common basis, is 1, and is expressed as "an annuity of 1." (Annuity tables are built on this basis.) Therefore, in order to find the present value of an annuity of 1, divide the compound discount by the rate per cent expressed decimally. Using the figures given above, the calculation would be: $.207904 \div .06 = 3.465105$, or the amount of an annuity of 1 for 4 periods at 6%.

Procedure: (*a*) Calculate the compound discount on 1 for the required number of periods and at the required rate per cent, $1 - v^n = D$.

(*b*) Divide the compound discount by the given rate per cent, expressed decimally. The quotient will be the present value of an annuity of 1, $D \div i = a_{\overline{n|}i}$.

(*c*) Multiply the present value of the annuity of 1 by the number of dollars of each rent, $R \times a_{\overline{n|}i} = A$.

Example

The terms of an annuity contract call for the payment of $100 at the end of each year for 4 years. If money is worth 6%, interest compounded annually, what is the present value of the annuity contract at the beginning of the first year?

Formula	*Arithmetical Substitution*
$Ra_{\overline{n|}i} = A$	$100\left[\dfrac{1 - \dfrac{1}{(1.06)^4}}{.06}\right] = \$346.51$

Solution

$(1.06)^4 = 1.262477$, compound amount of 1 for 4 years at 6%

$1 \div 1.262477 = .792094$, present value of 1 for 4 years at 6%

$1 - .792094 = .207906$, compound discount on 1 for 4 years at 6%

$.207906 \div .06 = 3.4651$, present value of an annuity of 1 for 4 years at 6%, or the value of $a_{\overline{4|}6\%}$.*

$3.4651 \times \$100 = \$346.51$, present value of an annuity of $100

Verification

Beginning of first year:		
Present value of contract.		$346 51
End of first year:		
Deduct:		
Rent.	$100.00	
Interest on $346.51 at 6%	20 79	
Reduction in value of annuity contract		79 21
Present value.		$267 30
End of second year:		
Deduct:		
Rent.	$100 00	
Interest on $267.30..	16 04	
Reduction in value of annuity contract....		83.96
Present value.		$183 34

* The value of $a_{\overline{4|}6\%}$ may be obtained directly from Table 5, page 532.

End of third year:		
Deduct:		
Rent.	$100.00	
Interest on $183.34	11.00	
Reduction in value of annuity contract.		89.00
Present value		$ 94 34
End of fourth year:		
Deduct:		
Rent.	$100 00	
Interest on $94.34	5 66	
Reduction in value of annuity contract		94 34
		$ 0 00

Amortization. Payments made on the principal, as shown in the above example, are known as amortization payments. Amortization is the gradual repayment of the principal through the operation of the two opposing forces of compound interest and periodic payments. Compound interest increases the principal, while the payments reduce it. In the verification above, it can be seen that the excess of each payment over the interest for the period is the amount by which the principal is reduced. The amount of this reduction is the amortization.

Problems

1. Find the present value of each of the following ordinary annuities:

	Rents	*Paid*	*Interest*	*Years*
(*a*)	$400	Annually	3%	6
(*b*)	$225	Annually	4%	10
(*c*)	$350	Annually	4%	10
(*d*)	$255	Semiannually	6%	3
(*e*)	$340	Semiannually	7%	3

2. By the terms of an annuity contract, $500 is to be paid at the end of each six months for 4 years. Money is worth 6%, interest compounded semiannually. (*a*) Find the present value of the annuity. (*b*) Submit solution and verification showing the applications each year of the payments as to interest, amortization of principal, and new principal against the present value of the annuity.

3. What is the value at the beginning of the first period of an annuity of $300 payable at the end of each year for 10 years, money being worth 4%, interest compounded semiannually? Prepare columnar table.

4. A man purchases a house for $1,800 cash and sixteen notes of $400 each, without interest, one due at the end of each six months until all the notes are paid. If money is worth 4%, interest compounded semiannually, what is the cash value of the property?

5. What is the cash value of a contract which calls for the payment of $50 at the end of each month for 5 years, if money is worth 6%, interest convertible monthly?

6. A disability insurance contract provides that the insured may choose one of the following options: (*a*) $50 a month, payable at the end of each month.

for 48 months; (*b*) \$500 cash, and \$50 a month for 36 months; (*c*) \$100 a month for 12 months, and \$50 a month for the ensuing 26 months. If money is worth 6%, interest compounded semiannually, which is the best option?

Computation of the rents or periodic payments of the present value of an ordinary annuity. If the present value of an annuity, the rate per cent, and the time are given, the rents or periodic payments may be calculated as follows:

Procedure: (*a*) Determine the present value of an annuity of 1 for the required number of periods at the given rate per period, or $a_{\overline{n}|i}$.

(*b*) Divide the given present value of the annuity by the present value of the annuity of 1 found in (*a*), $A \div a_{\overline{n}|i} = R$.

Example

What annual rent will be produced by an ordinary annuity the present value of which is \$346.51, if there are four rents, and money is worth 6%?

Formula

$$\frac{A}{a_{\overline{n}|i}} = R.$$

Arithmetical Substitution

$$\frac{346.51}{\dfrac{1 - \dfrac{1}{(1.06)^4}}{.06}} = \$100.00.$$

Solution

$(1.06)^4 = 1.262477$, compound amount of 1 for 4 years at 6%

$1 \div 1.262477 = .792094$, present value of 1 for 4 years at 6%

$1 - .792094 = .207906$, compound discount on 1 for 4 years at 6%

$.207906 \div .06 = 3.4651$, present value of an annuity of 1 for 4 years at 6%, or value of $a_{\overline{4}|6\%}$.*

$\$346.51 \div 3.4651 = \$100$, rent of annuity

See verification shown on pages 336–337.

Problems

1. If the present value of an annuity contract is \$6,000, what amount must be paid at the end of each year for 10 years to cancel the obligation, money being worth 4%, interest compounded annually?

2. An annuity contract is worth \$12,000 at the present date. If the time to maturity is 10 years, and money is worth 3%, interest compounded semiannually, what periodic payment must be made at the end of each six months to cancel the contract in 10 years?

3. The present value of a 12-year annuity contract is \$8,000. If money is worth 4%, interest convertible quarterly, what amount must be paid at the end of each quarter to cancel the contract in 12 years?

---

* This value is readily found in Table 5, page 532.

Payment of debt by installments. In many contracts it is agreed that the principal of the debt together with the interest are to be paid in equal periodic payments; each payment is to cancel the interest due to date, and the balance is to be applied toward the repayment of the principal.

The procedure, formula, and solution are similar to those given on page 338.

Example

If Smith borrows $1,000 from Jones at 6%, and agrees to cancel the debt (principal and interest) in five equal annual payments, what will be the amount of each payment?

The formula for "rent of present value of annuity," given on page 338, is applicable.

TABLE OF INSTALLMENT PAYMENTS

(1) *End of Period*	(2) *Payment Made*	(3) *To Cancel Interest*	(4) *Amortization of Principal*	(5) *Balance of Principal*
1	$ 237 39	$ 60 00	$177 39	$1,000 00
2	237 39	49 36	188 03	822 61
3	237 39	38 07	199 32	634 58
4	237 39	26 11	211 28	435 26
5	237 39	13 44	223 95	223 98
Totals	$1,186 95	$186 98	$999 97	$ 03

A slight error, such as that in the above table, is likely to occur in many problems in installment payments; it is caused by the fact that the computations are not carried to a sufficient number of decimal places. In such cases the difference should be corrected in the last payment. In the table, the fifth payment should be $237.42, instead of $237.39. This would leave column (5) with no remainder, and column (4) would have a total of $1,000.

Problems

1. *A* contracts for the purchase of a house, and agrees to pay for it in installments of equal amounts over a period of 10 years. The cash value of the house is $10,000. If money is worth 5%, interest convertible quarterly, what should be the amount of each payment?

2. What annuity, payable quarterly for 20 years, would be required to repay a loan of $12,840, the nominal rate of interest being 4% per annum?

3. A debt of $3,500, with interest at 5%, compounded semiannually, will be discharged, principal and interest, by equal payments at the end of each six months for 10 years. Determine the amount of each payment.

4. A house cost $15,000. The purchaser paid $3,000 cash, and agreed to pay the balance in equal quarterly payments, principal and interest, over a period of $8\frac{1}{2}$ years. If money is worth 6%, interest convertible quarterly, what is the amount of each payment?

5.* A company purchased machinery on December 1, at a cost of \$40,000. Twenty-five per cent of the cost was paid in cash, and the balance is to be paid in 60 monthly installments of equal amount. The monthly payments are to be represented by notes, payable on the first day of each month and secured by chattel mortgage. Interest at 6% per annum, or $\frac{1}{2}$ of 1% per month, is to be included in the notes. Compute the amount of each note, taking the compound interest on \$1 as .348850 for the given time and rate.

Computation of the term of an annuity. When the rate per cent, the amount of the annuity, and the size of each periodic payment are stated, it is possible to calculate the number of periodic payments to be made.

In calculating the time, it is necessary to resort to equations and logarithms.

Procedure: (*a*) Divide the amount of the annuity by the number of dollars in one of the periodic payments, to find the amount of the annuity of 1 at the given rate.

(*b*) Multiply the amount of the annuity of 1 by the rate per period expressed decimally, to find the compound interest on 1 for the unknown time.

(*c*) Add 1 to the compound interest on 1 to find the compound amount of 1.

(*d*) Determine the log of the compound amount of 1 found in (*c*).

(*e*) Determine the log of 1 plus the rate per cent for the period expressed decimally; that is, the ratio of increase.

(*f*) Divide the log of the compound amount of 1, (*d*), by the log of the ratio of increase, (*e*), to find the number of periods, or the number of periodic payments.

Example

The amount of an annuity is \$1,318.08, the rent is \$100 each year, and the rate is 6%. Find the term.

Formula

$$\frac{\log\left[1 + \left(\dfrac{\text{Amount of annuity}}{\text{Rent}} \times \text{Rate}\right)\right]}{\log\ (\text{Ratio of increase})} = \text{Term}$$

Arithmetical Substitution

$$\frac{\log\left[1 + \left(\dfrac{1{,}318.08}{100} \times .06\right)\right]}{\log\ (1.06)} = 10$$

Solution

Dividing by rent:	$1{,}318.08 \div 100 = 13.1808$
Multiplying by rate:	$13.1808 \times .06 = .790848$

* C. P. A., Ohio.

Adding 1:	$1 + .790848 = 1.790848$
log of:	$1.790848 = 0.253059$
log of:	$1.06 = 0.025306$
Dividing:	$0.253059 \div 0.025306 = 10$

As the rents are to be paid annually, the annuity term will be 10 years.

If the payments are to be made more often than once each year, the rate should be reduced to a rate per period.

Example

A purchases a house for \$2,629.02, and agrees to pay \$500 down and \$50 at the end of each month. Each payment is to cancel the interest to date, and the balance is to apply against the principal. The debt bears 6% interest. How many months will it take A to pay off the debt?

Reducing the rate to the rate per period, .06 ÷ 12 = .005.

Formula — *Arithmetical Substitution*

$$\frac{\log\left[\dfrac{1}{1-\left(\dfrac{\text{Debt}}{\text{Rent}}\times i\right)}\right]}{\log\,(1+i)} = \text{Term} \qquad \frac{\log\left[\dfrac{1}{1-\left(\dfrac{2{,}129.02}{50}\times .005\right)}\right]}{\log\,1.005} = 48$$

Solution

Dividing:	$2{,}129.02 \div 50 = 42.5804$
Multiplying:	$42.5804 \times .005 = .212902$
Subtracting:	$1 - .212902 = .787098$
Dividing:	$1 \div .787098 = 1.270489$
log of:	$1.270489 = 0.1039709$
log of:	$1.005 = 0.0021661$
Dividing:	$0.1039709 \div 0.0021661 = 48$, approx.

Hence 48 monthly payments will be necessary to pay off the debt.

Problems

1. L. Miller purchased a house and lot for \$7,800. He agreed to pay \$2,800 cash, and \$50 at the end of each month until the debt should be paid. How many months will it take Miller to pay the debt, if each payment is to cancel the interest due and the balance is to apply against the principal? The interest rate is 6%.

2. B sells his residence for \$10,500. He receives a down payment of \$2,500. The balance is to be paid on the basis of \$75 each month. Each monthly payment is to cancel the interest first, and the balance is to apply against the principal. The contract states that interest is at the rate of 6% per annum. How long will it take the buyer to pay off the debt?

3. Cole buys a farm for \$18,000, and agrees to pay \$6,000 down, the balance to draw interest at 6% until paid. Any payments made are to apply on interest due to date, and if the payments exceed the interest due, the balance is to be applied to the reduction of the principal. Cole desires to know how long it will take him to pay for the farm if he makes equal quarterly payments of \$400.

Use of effective rate in annuities. If interest is compounded more frequently than the rents are paid, or vice versa, it is neces-

sary to reduce the rate of interest to an effective rate for a period corresponding to the periods of the rent payments.

Procedure: (*a*) Calculate the effective rate of interest for one rent period.

(*b*.-1) If the amount of the annuity is known, use the procedure previously given.

(*b*.-2) If the present value of the annuity is known, use the procedure previously given.

Example

The rents of \$100 each are to be paid annually, the amount is \$318.64, and the interest rate is 6%, compounded semiannually. Find the term.

PART 1

Formula	*Arithmetical Substitution*
$\left(1+\frac{j}{m}\right)^m - 1 = \text{Effective rate}$	$\left(1+\frac{.06}{2}\right)^2 - 1 = .0609$

Solution

$$\left(1+\frac{.06}{2}\right)^2 = 1.0609\text{, effective ratio of increase for 1 year}$$

$$1.0609 - 1 = .0609\text{, effective rate}$$

PART 2

As the problem states the amount of the annuity, it may be solved by procedure (*b*.-1).

Formula	*Arithmetical Substitution*
$\dfrac{\log\left[1+\left(\dfrac{\text{Amount}}{\text{Rent}}\times i\right)\right]}{\log(\text{Ratio of increase})} = \text{Term}$	$\dfrac{\log\left[1+\left(\dfrac{318.64}{100}\times .0609\right)\right]}{\log 1.0609} = 3$

Solution

Dividing:	$318.64 \div 100 = 3.1864$
Multiplying:	$3.1864 \times .0609 = .1940517$
Adding 1:	$1 + .1940517 = 1.1940517$
log of:	$1.1940517 = 0.077022$
log of:	$1.0609 = 0.025674$
Dividing:	$\log 0.077022 \div \log 0.025674 = 3$

The result, 3, indicates that there are three annual payments of \$100 each.

Verification

First year:		
Rent		\$100 00
Second year:		
\$100 × .0609	\$ 6 09	
Rent	100 00	106 09
		\$206 09
Third year:		
\$206.09 × .0609	\$ 12 55	
Rent	100 00	112 55
Amount		\$318 64

Problems

1. Compute the number of periods in each of the following:

	Amount of an Ordinary Annuity	*Rents*	*Rate*
(*a*)	\$ 8 0191	\$ 1 00	$4\frac{1}{2}$%
(*b*)	663.29	100 00	4%
(*c*)	9,549.11	1,000 00	5%
(*d*)	1,061.82	200 00	3%

2. Smith desires to repay his debt of \$5,000 by paying \$500 at the end of each year. If money is worth 5%, interest convertible semiannually, how many payments will he have to make?

3. *A* has a debt of \$10,570. He makes a payment of \$1,000 at the end of each year. Money is worth 6%, interest convertible quarterly. How many full periodic payments will *A* have to make in order to cancel the debt?

4. To repay a loan of \$1,000 bearing interest at 4%, convertible quarterly, Adams makes a payment of \$100 at the end of each six months. How long will it take him to cancel the debt?

5. A building is purchased for \$15,000. Payment is to be made in installments of \$1,500 at the end of each six months' period. Interest is at 6%, compounded quarterly. How many payments will be required to cancel the debt?

Computation of the rate of an annuity. The mathematical theory of an annuity deals with five elements term, rent, rate of interest, present value of the annuity, and amount of the annuity. If the rate and any other three elements are given, the missing element may be found. However, as the rate is used twice in annuity calculations, it can be only approximated if it is not given.

To obtain an approximate rate by inspection, find a trial rate, and test it to see whether, when it is used with the given component parts, it produces an amount equivalent to or nearly equivalent to the amount of the annuity. If the trial rate proves to be near the required rate, use it as a basis, and select another rate such that of the two rates chosen one will be more and the other less than the required rate. Proceed to select the approximate rate by the process of interpolation.

The test rates chosen may be found by means of an annuity table, or by the calculation of the amount of an annuity of 1.

Selection of rates by use of an annuity table.

Procedure: (*a*) Divide the given amount of the annuity by the annuity rent, to find the amount of an annuity of 1.

(*b*) Choose from the annuity table the two amounts nearest to the amount found in (*a*).

(*c*) Proceed by interpolation to find the approximate required rate.

Example

An ordinary annuity the amount of which is $1,099.62 has five annual rents of $200 each. What is the rate of the annuity?

SECTION OF ANNUITY TABLE

Periods	*3½%*	*4%*	*4½%*	*5%*	*6%*
1	1.00000	1.00000	1.00000	1.00000	1.00000
2	2.03500	2.04000	2.04500	2.05000	2.06000
3	3.10622	3.12160	3.13702	3.15250	3.18360
4	4.21494	4.24646	4.27819	4.31012	4.37461
5	5.36246	5.41632	5.47071	5.52563	5.63709

Solution

$1,099.62 ÷ $200 = $5.4981, amount of an annuity of $1 for 5 periods

The second step is to choose from the annuity table, horizontally to the right of the fifth period, the two amounts nearest to $5.4981. By inspection, the amount $5.47071, in the 4½% column, is found to be the nearest one under $5.4981, and $5.52563, in the 5% column, is found to be the nearest one over $5.4981. Using these rates and amounts as a basis, the following may be derived:

Amount of a $1 annuity at 5%	$5.52563
Amount of a $1 annuity at 4½%	5.47071
Difference in amount caused by ½% difference in interest rate	$ .05492
Amount of a $1 annuity at unknown rate	$5.4981
Less amount at 4½%	5.47071
Difference	$ .02739

$$\frac{2739}{5492} \text{ of } \tfrac{1}{2}\% = .2493\%, \text{ or approximately } \tfrac{1}{4}\%$$

$$4\tfrac{1}{2}\% + \tfrac{1}{4}\% = 4\tfrac{3}{4}\%, \text{ the approximate rate}$$

Selection of rate by calculation of amounts of annuities. If no annuity table is at hand, it is necessary to obtain the two basic rates by the calculation of the amounts of the annuities, using estimated rates.

Procedure: (*a*) As this is a five-payment annuity and each payment is $200, the total paid in will be $1,000. Deducting this $1,000 from the amount, $1,099.62, the interest is found to be $99.62.

(*b*) The payments will draw interest thus:

$200 for 4 years
200 for 3 years
200 for 2 years
200 for 1 year
200 for 0 years
10 years

Hence we have the equivalent of $200 for 10 years.

Simple interest on $200 for 10 years at 5% is $100, or a little more than the interest in the problem when compound interest is disregarded. Thus, it can be seen that the interest is probably less than 5%, but as the compounding of interest is infrequent, the variation will be small. Hence it is advisable to try the rate of 5%.

First trial rate. The solution by the 5% rate is as follows:

$$200\left(\frac{(1.05)^5 - 1}{.05}\right) = \$1{,}105.13.$$

Solution

$(1.05)^5 = 1.276281$, compound amount of 1 at 5% for 5 periods
$1.276281 - 1 = .276281$, compound interest on 1 at 5% for 5 periods
$.276281 \div .05 = 5.52563$, amount of annuity of 1 at 5% for 5 periods
$\$200 \times 5.52563 = \$1{,}105.13$, amount of annuity of $200 at 5% for 5 periods

This is found to be a little more than the required amount; therefore the next trial rate should be less than 5%.

Second trial rate. A trial rate of 4½% will be used.

$$200\left(\frac{(1.045)^5 - 1}{.045}\right) = \$1{,}094.14.$$

Solution

$(1.045)^5 = 1.2461819$, compound amount of 1 at 4.5% for 5 years
$1.2461819 - 1 = .2461819$, compound interest on 1 at 4.5% for 5 years
$2461819 \div .045 = 5.470709$, amount of annuity of 1 at 4.5% for 5 years
$\$200 \times 5.470709 = \$1{,}094.14$, amount of annuity of $200 at 4.5% for 5 years

This is found to be smaller than the amount in the problem, $1,099.62. Interpolate between the rates used in the first and second trials, as follows:

Interpolation of Amounts

Amount of annuity at 5%	$1,105.13
Amount of annuity at 4½%	1,094 14
Difference in amount caused by ½% difference in rate	$ 10 99
Amount of annuity at unknown rate	$1,099 62
Amount of annuity at 4½%	1,094.14
Difference in amount	$ 5 48

$\frac{548}{1099}$ of ½% = approximately ¼%
4½% + ¼% = 4¾%, the required rate

The approximate rate is found from the present value of annuities in exactly the same manner as from the amounts of annuities, except that the formula for the present value is used instead of the formula for the amounts.

Problems

1. Calculate the rates in each of the following:

No.	*Amount of Annuity*	*Rents*	*Periods*
(*a*)	\$ 5 58	\$ 1 00	5
(*b*)	232 76	10 00	15
(*c*)	4,486 52	100 00	21
(*d*)	4,358 54	125 00	20
(*e*)	1,729 46	137 50	10
(*f*)	347 50	20 00	12
(*g*)	6,684 00	200 00	20

2. Calculate the rates in each of the following:

No.	*Present Value of Annuity*	*Rents*	*Periods*
(*a*)	\$ 4 45	\$ 1 00	5
(*b*)	77 22	10 00	10
(*c*)	196 90	25 50	10
(*d*)	5,018 75	500 00	15
(*e*)	657.42	20 00	36 monthly payments.
(*f*)	1,200 00	100 00	20
(*g*)	7,500 00	500 00	25

Solution of annuity problem with limited data. It is not uncommon to find in examinations in accounting a problem followed by a list of numerical values of certain terms, from which the candidate must calculate certain other values needed for solution of the problem. The purpose of this is to test the candidate's knowledge of the relationships between actuarial terms.

Problems

1.* A company is issuing \$100,000 of 4%, 20-year bonds, which are to be paid at maturity by means of a sinking fund into which annual deposits are to be made. The board of directors wishes to assume that this fund will earn $5\frac{1}{2}\%$ for the first 5 years, 5% for the next 5 years, and 4% for the last 10 years. What is the annual deposit required?

Given:

	$5\frac{1}{2}\%$	5%	4%
S_5	5 581	5 526	5 416
S_{10}	12.875	12.578	12 006
$(1+i)^5$	1.307	1.276	1 217
$(1+i)^{10}$	1.708	1.629	1.480

* American Institute Examination.

2.* A city, with its fiscal year ending April 30, prepares its budget and makes its tax levy for the subsequent fiscal year during March, taxes being payable on or after November 1.

A bond election was held in June, 1942, and bonds of \$1,000,000 were issued dated August 1, 1942, due in 20 years. A sinking fund was to be provided, calculated on a basis of 4%, interest compounded annually.

An audit having been made as of April 30, 1943, the balance of \$409,588.25 in the sinking fund is found to differ from the actuarial requirements.

Calculate the correct amount which should have been in the fund, and ascertain the annual adjustment that the city must thereafter make to be able to meet the bonds at maturity; the difference is to be spread over the subsequent levies, and not provided for in the next levy only.

Assume that 4% interest will be earned in the future, that all taxes will be collected in full by the end of the fiscal year, and that a deposit of the correct amount is to be made in the sinking fund annually on April 30.

Given, at 4%:

$v^8 = .7306902$	$(1+i)^8 = 1.3685690$
$v^9 = .7205867$	$(1+i)^9 = 1.4233118$
$v^{10} = .6755642$	$(1+i)^{10} = 1.4802443$
	$(1+i)^{19} = 2.1068492$
	$(1+i)^{20} = 2.1911231$
	$(1+i)^{21} = 2.2787681$

Review Problems

1. Amos Brown sets aside \$400 at the end of each year to provide a fund for his daughter's college expenses. If he invests the money at 3% effective, compounded annually, what will be the amount at the end of 10 years?

2. Ben Told invests \$200 at the end of each year. At the end of the fifth year he has accumulated \$1,040.81. Write the equation whose solution will give the rate of interest. Solve and check your answer as nearly as possible from the $s_{\overline{n}|i}$ table. Interest convertible annually.

3. If in Problem 1 the interest realized had been $2\frac{1}{2}$%, what would have been the amount?

4. An annuity of \$1.00 a year amounted to \$8.00 in 7 years. What was the effective rate of interest?

5. If money can be invested at 3% effective, how many full years will be necessary to accumulate a fund of at least \$2,000 from \$100 set aside at the end of each year?

6. Find the amount of an annuity of \$1,000 a year paid in four quarterly installments of \$250 for 6 years if the interest rate is 4% effective.

7. How long will it take to accumulate \$1,500 by depositing \$20 at the end of each month if the bank pays 2% effective? Give your answer to the nearest month.

8. If you pay a paving tax of \$52.17 at the end of each year for 10 years and the rate of interest is 5%, what is the actual tax for the paving?

9. What is the present value of an annuity of \$1,800 a year in monthly installments for 10 years if money is worth 4% effective?

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* American Institute Examination.

10. How long will it take to pay for a house and lot priced at \$6,000 if you pay \$1,000 down and \$800 at the end of each year until full payment is made, assuming interest to be 6% effective?

11. If \$750 invested at the end of each year for 6 years amounts to \$4,912.62, what is the rate of interest?

12. An annuity of \$100 a year for 8 years amounts to \$900. Find the effective rate of interest, convertible annually.

13. Find the amount of an annuity of \$500 for 10 years: (*a*) with effective rate 4%; (*b*) with a nominal rate of 4%, converted quarterly.

14. George Smith deposits \$50 in a savings bank at the end of each three months. The bank pays 2% convertible semiannually. What will be the amount to Smith's credit at the end of 5 years?

15. Find the present value of an annuity of \$500 a year for 10 years if money is worth 4% effective.

16. A realtor offers a house for \$4,000 cash and \$1,000 a year for six years without interest. A buyer desires to pay cash. If money is worth 6% effective, what should be the cash price of the house?

17. Find the present value of an annuity of \$2,000 a year for 5 years if money is worth 4%, converted quarterly.

18. A realtor offers a house for \$1,500 cash and \$50 a month for 10 years, without interest. If money is worth 6% effective, what is the equivalent cash price?

19. What is the present value of an annuity of \$840 a year in quarterly installments for six years: (*a*) if money is worth 4% nominal, convertible quarterly; (*b*) if money is worth 4% nominal, convertible semiannually?

20. How many years will it take to accumulate \$785 if \$100 is invested at the end of each year at $4\frac{1}{2}$%?

CHAPTER 32

Special Annuities

Annuity due. An annuity due is an annuity the periodic payments of which are made at the begining of each period. A comparison of the following tables will show the difference between an annuity due and an ordinary annuity; it will be remembered that the payments of an ordinary annuity are made at the end of each period.

(1)	(2)	(3) *Rents Payable in Ordinary Annuity*	(4) *Rents Payable in Annuity Due*
Beginning of 1st yr.	Contract made	None	1st payment
" " 2nd "		1st payment	2nd "
" " 3rd "		2nd "	3rd "
" " 4th "		3rd "	4th "
" " 5th "	Contract ends	4th "	None

(1) *End of Period*	(2) *4-Period Compound Amount*	(3) *Amount of Ordinary Annuity*	(4) *5-Period Compound Amount*	(5) *Amount of Annuity Due*
1	1 00	1 00	1 00	1 06
2	1 06	2 06	1 06	2 1836
3	1.1236	3 1836	1.1236	3 374616
4	1 191016	4 374616	1 191016	4 637093
5	1 262477	5 637093	1 262477	
			5 637093	

Amount of an ordinary annuity for 5 periods...	5 637093
Deduct	1 000000
Amount of an annuity due for 4 periods .	4 637093

To find the amount of an annuity due. From the above tabulations it can be seen that the amount of an annuity due of 1 for 4 periods is 1 less than the amount of an ordinary annuity for 5 periods.

ANALYSIS OF THE AMOUNT OF AN ANNUITY DUE

Assumed Periods	*First Period*	*Second Period*	*Third Period*	*Fourth Period*	*Total Value, End of Last Period*
Rent, beginning of 1st yr	1	....	....	....	1 262477
" " " 2nd "		1	....	....	1 191016
" " " 3rd "			1	....	1 1236
" " " 4th "				1	1 06
End of 4th year, annuity due					4 637093

As shown above, each periodic rent draws interest from the begining of the year in which it is deposited, and continues to draw interest until the due date. The sum of these periodic amounts is the amount of the annuity due.

Since each payment of an annuity due is made at the beginning of a period, the final result is that the amount of an annuity due exceeds the amount of an ordinary annuity by the interest on the amount of the ordinary annuity for one period.

The amount of an annuity due, represented by the symbol S', may be found in either of two ways:

(1) By adding the interest for one period to the amount of an ordinary annuity for the number of periods specified.

(2) By finding the amount of an ordinary annuity for one payment more than the number of payments specified, and then deducting one payment or rent from the total.

First method. Procedure: (*a*) Compute the amount of an ordinary annuity at the given rate per cent and for the given rents.

(*b*) Multiply the amount found in (*a*) by 1 plus the rate of interest per period expressed decimally.

Example

Annuity payments of $100 are to be made at the beginning of each year for 4 years. Money is worth 6%. What is the amount of the annuity?

Formula

$$(Rs_{\overline{n}|i})(1 + i) = S'$$

Arithmetical Substitution

$$\left[100\left(\frac{(1.06)^4 - 1}{.06}\right)\right](1.06) = \$463.71.$$

Solution

$(1.06)^4 = 1.262477$, compound amount of 1 at 6% for 4 periods

$1.262477 - 1 = .262477$, compound interest on 1 at 6% for 4 periods

$.262477 \div .06 = 4.374616$, amount of an ordinary annuity of 1 at 6% for 4 periods, or $s_{\overline{4}|6\%}$.*

* The value of $s_{\overline{4}|6\%}$ may be found in Table 4, page 529.

100 × 4.374616 = 437.4616, amount of an ordinary annuity of 100 at 6% for 4 periods
$437.4616 × 1.06 = $463.71, amount of an annuity due of $100

Verification

Beginning of 1st period:		
Contract made.		
Rent	$100.00	$100 00
Beginning of 2nd period:		
Rent	100 00	
Interest on $100 at 6%	6 00	106.00
New principal		$206 00
Beginning of 3rd period:		
Rent	$100 00	
Interest on $206 at 6%	12 36	112 36
New principal		$318.36
Beginning of 4th period:		
Rent	$100 00	
Interest on $318.36 at 6%	19 10	119 10
New principal		$437 46
End of 4th period:		
Interest on $437.46		26 25
Amount due		$463 71

Second method. Procedure: (*a*) Determine the amount of an ordinary annuity of 1 for the required number of periods plus 1.

(*b*) Deduct 1 from the result found in (*a*).

(*c*) Multiply the difference found in (*b*) by the number of dollars in each periodic rent, and this product will be the amount of the annuity due.

Formula

$$R(s_{\overline{n+1}|i} - 1) = S'$$

Arithmetical Substitution

$$100\left(\frac{(1.06)^{4+1} - 1}{.06} - 1\right) = \$463.71.$$

Solution

$(1.06)^5$ = 1.338225, compound amount of 1 for 5 periods at 6%
1.338225 − 1 = .338225, compound interest on 1 for 5 periods at 6%
.338225 ÷ .06 = 5.63709, amount of an ordinary annuity of 1 for 5 periods at 6%, or $s_{\overline{5}|6\%}$ from Table 4, page 529.
5.63709 − 1 = 4.63709, amount of an annuity due of 1 for 4 periods at 6%
$100 × 4.63709 = $463.71, amount of an annuity due of $100 for 4 years at 6%

Problems

Prepare the formula, solution, and verification for each of the following:

1. An annuity contract calls for the payment of $1,000 at the beginning of each year for 5 years. Money is worth 6%, interest compounded annually. What is the amount of the annuity at the end of the fifth year?

2. For 5 years a man deposits in the bank $150 on the first of each quarter. The bank allows him 4% interest, compounded quarterly. What will be the amount of his savings in the bank at the end of the 5-year period?

3. The X.Y.Z. Company deeds a house and lot to Smith. In return, Smith is to deposit with the company, over a period of 10 years, $200 at the beginning of each six months. The rate of interest is to be 6%, compounded semiannually. What will be the amount of the accumulation at the end of the 10-year period?

Present value of an annuity due. The present value of an annuity due, represented by the symbol A', is the present or actual cash value, at the date of the first payment, of all the payments to be made under the annuity contract. The following table shows the present value of an annuity due for which payments of 1 are to be made for 4 years; interest is calculated at 6%, compounded annually:

	Value at Beginning of Period	*First Period*	*Second Period*	*Third Period*	*Fourth Period*
Rent, beginning of first period..	1 000000	1			
" " " " "	943396		1.		
" " " " "	889996			1.	
" " " " "	.839619				1
Total.	3 673011				

Comparison of present value of an ordinary annuity and that of an annuity due.

Example

Under the terms of an annuity due, four annual payments of $1 each are to be made. Interest is at 6%. Find the present value of the annuity.

TABLE OF COMPARISON OF PRESENT VALUE OF AN ORDINARY ANNUITY AND AN ANNUITY DUE

(1) *Number of Rents*	(2) *Four Rents, Ordinary Annuity*	(3) *Three Rents, Ordinary Annuity*	(4) *Annuity Due of Four Rents*
1	.943396	943396	1 000000
2	1 833392	1 833392	1 943396
3	2 673011	2 673011	2 833392
4	3 465105		3 673011

To find the present value of an annuity due. This may be done in two ways:

(1) If a number in column 2 of the above table is multiplied by 1 plus the rate per cent, (1.06), the product will be the number in column 4 corresponding to the same number of rents; thus: $3.465105 \times 1.06 = 3.673011$, the present value of an annuity due at 6% for 4 periods.

(2) If 1 is added to the present value of an ordinary annuity, the sum will be the present value of an annuity due of one more rent than the ordinary annuity; thus:

Present value of an ordinary annuity of 3 rents........	2.673011
Adding 1	1 000000
Present value of an annuity due for 4 periods..........	3 673011

First method. Procedure: (*a*) Compute the present value of an ordinary annuity, using the given number of dollars in the rents, the given rate per cent, and the given number of periods.

(*b*) Multiply the present value found in (*a*) by the rate per cent per period plus 1.

Example

Under the terms of an annuity due, four annual payments of $100 each are to be made. Money is worth 6%, interest compounded annually. Find the present value of the annuity.

Formula

$$(Ra_{\overline{n}|i})(1 + i) = A'$$

Arithmetical Substitution

$$100\left(\frac{1 - \dfrac{1}{(1.06)^4}}{.06}\right)(1.06) = \$367.30.$$

Solution

$(1.06)^4 = 1.262477$, compound amount of 1 at 6% for 4 periods

$1 \div 1.262477 = .7920937$, present value of 1 at 6% for 4 periods

$1 - .7920937 = .2079063$, compound discount on 1 at 6% for 4 periods

$.2079063 \div .06 = 3.465105$, present value of an ordinary annuity of 1, or $a_{\overline{4}|6\%}$ from Table 5, page 532.

$100 \times 3.465105 = 346.5105$, present value of an ordinary annuity of 100

$\$346.5105 \times 1.06 = \$367.30$, present value of an annuity due of $100

Verification

Beginning of first period:		
Present value		$367.30
Rent deducted	$100.00	100.00
New principal		$267.30
Beginning of second period:		
Rent	100.00	
Less interest on $267.30	16 04	
Balance to apply on principal		83.96
New principal		$183.34
Beginning of third period:		
Rent	$100.00	
Less interest on $183.34	11 00	
Balance to apply on principal		89 00
New principal		$94.34
Beginning of fourth period:		
Rent	$100 00	
Less interest on $94.34	5 66	
Balance to apply on principal		94 34

Second method. Procedure: (*a*) Compute the present value of an ordinary annuity of 1 at the required rate and for one less than the required number of periods.

(*b*) To the present value of the annuity found in (*a*) add 1, and the result is the present value of an annuity due of 1 for the required number of periods.

(*c*) Multiply the present value of an annuity due of 1 by the number of dollars in each rent.

Formula

$$R(a_{\overline{n-1}|i} + 1) = A'$$

Arithmetical Substitution

$$100\left(\frac{1 - \dfrac{1}{(1.06)^{4-1}}}{.06} + 1\right) = \$367.30.$$

Solution

$(1.06)^{4-1}$, or $(1.06)^3$ = 1.191016, compound amount of 1 for 3 periods at 6%

$1 \div 1.191016$ = .839619, present value of 1 for 3 periods at 6%

$1 - .839619$ = .160381, compound discount on 1 for 3 periods at 6%

$.160381 \div .06$ = 2.6730, present value of an ordinary annuity of 1 for 3 periods at 6%, or $a_{\overline{3}|6\%}$ from Table 5, page 532.

$2.6730 + 1$ = 3.6730, present value of an annuity due of 1 for 4 periods at 6%

$\$100 \times 3.6730$ = $367.30, present value of an annuity due of $100 for 4 years at 6%

Problems

1. What is the present value of an annuity due in which $100 payments are to be made on the first day of each six months for 10 years, if money is worth 5%, interest compounded semiannually? Prepare formula, solution, and verification.

2. The rents of an annuity due are $500 each, and are payable semiannually for 10 years. If money is worth 6%, interest compounded semiannually, what is the value of the annuity at the date of the payment of the first rent? Prepare formula, solution, and verification.

3. A contract provides for the payment of $150 on the first of each quarter for a period of 10 years. Interest is 4%, compounded quarterly. What is the present value of the contract?

4. What is the present value of a contract which calls for the payment of $50 on the first of each month for a period of 10 years, interest to be computed monthly at 6% per annum?

5.* *A* is considering two propositions for the investment of $75,000 belonging to an estate. The first proposition offers him six 7% notes maturing as follows:

On July 1, 1943	$ 5,000
On July 1, 1945	5,000
On July 1, 1947	5,000
On July 1, 1949	5,000
On July 1, 1950	5,000
On July 1, 1951	50,000
Total	$75,000

The second proposition offers him two 5% notes, maturing as follows:

On July 1, 1946	$25,000
On July 1, 1951	60,000
Total	$85,000

In each case the loan is adequately secured, and the interest is payable semiannually; each proposition is offered to *A* for $75,000 in cash on July 1, 1941.

A requests you to determine which proposition is the better one for him to accept. State your findings, and demonstrate the correctness of your answer.

Given: The present value of 1, ten periods hence at $3\frac{1}{2}$%, is .708919.

Rents of the amount of an annuity due. The rents may be found by the following procedure:

Procedure: (*a*) Use the Second Method (see page 351) to compute the amount of an annuity due of 1 for the required number of periods and at the required rate per period.

(*b*) Divide the given amount of the annuity due by the amount of an annuity due of 1, as computed in (*a*), to find the rent.

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* Adapted from C. P. A., Illinois.

Example

The rents of an annuity due are paid at the beginning of each year for 4 years. At the end of the fourth year the amount of the annuity is $100. Interest is at 6%, compounded annually. Compute the rents.

Formula

$$\frac{P}{s_{\overline{n+1}|i} - 1} = R'$$

Arithmetical Substitution

$$\frac{100}{\dfrac{(1.06)^{4+1} - 1}{.06} - 1} = \$21.57.$$

Solution

$(1.06)^5 = 1.338225$, compound amount of 1 for 5 periods at 6%
$1.338225 - 1 = .338225$, compound interest on 1 for 5 periods at 6%
$.338225 \div .06 = 5.6371$, amount of an ordinary annuity of 1 for 5 periods at 6%, or $s_{\overline{5}|6\%}$ from Table 4, page 529
$5.6371 - 1 = 4.6371$, amount of an annuity due of 1 for 4 periods at 6%
$\$100 \div 4.6371 = \$21.57$, rent of the amount of an annuity due of $100 for 4 years at 6%

Verification

Beginning of first year:		
Rent		$ 21 57
Beginning of second year:		
Rent	$21 .57	
Interest on $21.57 at 6%	1.29	22 86
Amount		$ 44 43
Beginning of third year:		
Rent	$21 .57	
Interest on $44.43 at 6%	2.67	24 24
Amount		$ 68.67
Beginning of fourth year:		
Rent	$21 57	
Interest on $68.67 at 6%	4.12	25 69
Amount		$ 94 36
End of fourth year:		
Interest on $94.36 at 6%		5 66
		$100 02

Problems

1. What are the rents of an annuity due which amount to $4,762.40 in 10 years, if money is worth 6% per annum? Prepare formula, solution, and verification.

2. Mr. Ames desires to know how much he must deposit in the bank on the first day of each six months' period for 10 years, in order that at the end of the

tenth year he may have \$12,000 accumulated. The bank pays 4%, interest compounded semiannually. What is the semiannual deposit required? Prepare formula, solution, and verification.

3. Brown wishes to save \$8,000 by making equal deposits on the first of each quarter for 10 years. His bank allows him 4%, interest compounded quarterly. What is the quarterly deposit required?

4. A wishes to have \$5,000 saved at the end of 3 years. He decides to make equal deposits on the first of each month. His bank allows him 6%, interest compounded monthly. What is the monthly deposit required?

Rent of the present value of an annuity due. The rent of the present value of an annuity due may be found by the following procedure:

Procedure: (*a*) Use the Second Method (see page 354) to compute the present value of an annuity due of 1 for the required number of periods and at the given rate per cent.

(*b*) Divide the given present value of the annuity due by the present value of an annuity of 1, as computed in (*a*).

Example

A man owes \$5,000. He wishes to know the size of each of ten equal annual payments which will exactly cancel the debt, and pay the accrued interest due on the date of each payment. The first payment is to be made at once. Money is worth 6% per annum.

Formula

$$\frac{P}{a_{\overline{n-1}|i} + 1} = R'$$

Arithmetical Substitution

$$\frac{5{,}000}{\left(\dfrac{1 - \dfrac{1}{1.06)^{10-1}}}{.06} + 1\right)} = \$640.89.$$

Solution

$(1.06)^9 = 1.689479$, compound amount of 1 for 9 periods at 6%

$1 \div 1.689479 = .591898$, present value of 1 for 9 periods at 6%

$1 - .591898 = .408102$, compound discount on 1 for 9 periods at 6%

$.408102 \div .06 = 6.8017$, present value of an ordinary annuity for 9 periods at 6%, or $a_{\overline{9}|6\%}$ from Table 5, page 532.

$6.8017 + 1 = 7.8017$, present value of an annuity due for 10 periods at 6%

$\$5{,}000 \div 7.8017 = \$640.89$, present value of an annuity due of \$5,000

Problems

1. Give the formula, solution, and verification for each of the following:

	Debt to Be Retired	Payments	Rents	Per Cent	Compounded
(a)	$1,000 00	4	....	6	Annually
(b)	2,762 50	8	....	4	Semiannually
(c)	4,875 40	16	....	6	Quarterly

2. *A* purchases a farm for $15,000, with interest at 6%. He desires to pay for it in twelve equal annual payments, the first payment to be made immediately. What is the amount of each payment necessary to cancel the debt and interest?

Effective interest on an annuity due. When the interest is compounded more often than the payments are made, the effective rate should be found for a period corresponding to the period of an annuity payment.

Procedure: (*a*) Reduce the given interest rate to an effective rate for a period corresponding to the period of an annuity payment.

(*b*) Using as the rate for each period the effective rate found in (*a*), proceed by following the instructions given in preceding pages of this chapter.

Example

What is the amount of an annuity due, the annual payments of which are $100 for 3 years, and the interest on which is 6%, compounded quarterly?

PART 1

Formula — *Arithmetical Substitution*

$$\left(1+\frac{j}{m}\right)^n - 1 = \text{Effective rate} \qquad \left(1+\frac{.06}{4}\right)^4 - 1 = .0613635$$

Solution to Part 1

$.06 \div 4 = .015$, quarterly rate
$1 + .015 = 1.015$, ratio of increase for 1 quarter
$(1.015)^4 = 1.0613635$, compound amount of 1 for 4 periods
$1.0613635 - 1 = .0613635$, effective interest for an annuity payment period

PART 2

Formula

$$R\left(\frac{(1+i)^{n+1}-1}{i} - 1\right) = \text{Amount of annuity due}$$

Arithmetical Substitution

$$100\left(\frac{(1.0613635)^4 - 1}{.0613635} - 1\right) = \$338.34$$

Solution to Part 2

Substituting for i the rate found in Part 1, the solution is:

$(1.0613635)^4 = 1.2689855$, compound amount of 1 for 4 periods at 6.13635%

$1.2689855 - 1 = .2689855$, compound interest on 1 for 4 periods at 6.13635%

$.2689855 \div .0613635 = 4.3834$, amount of an ordinary annuity of 1 for 4 periods at 6.13635%

$4.3834 - 1 = 3.3834$, amount of an annuity due for 3 periods at 6.13635%

$\$100 \times 3.3834 = \$338.34$, amount of annuity due of \$100 for 3 years

Problems

1. What will be the amount at the end of 4 years of an annuity the rents of which are \$1,000 payable at the end of each year, and the nominal rate 4%, interest compounded semiannually? Give formula, solution, and verification.

2. A company desires to know how much will be accumulated at the end of 5 years if \$5,000 is placed in a sinking fund at the end of each year. The fund bears a nominal rate of 6%, interest compounded semiannually. Give verification.

3. The R-M Company desires to accumulate a sinking fund of \$50,000 to meet a bond issue of that amount which is due in 10 years. What will be the annual payments made on the last day of each year, if the fund is so placed that it will bear an annual rate of 6%, interest to be compounded semiannually? Construct columnar table showing payments, interest, addition to fund, and amount of fund.

4. The R.P.S. Company has a bond issue of \$50,000 coming due at the end of 5 years. The directors desire to know how much money must be placed in the bank at the end of each year in order that this fund may exactly cover the bond issue at the end of the required time. The bank pays 4%, interest compounded quarterly. Construct a columnar table, as in the preceding problem.

5. The annual rents of an ordinary annuity are \$500, the time is 6 years, and money is worth 6%, interest compounded semiannually; what is the present value? Construct a columnar table.

6. What is the present value of a contract in which a company agrees to pay \$500 at the beginning of each year for 5 years, interest to be allowed at the rate of 5% per annum, compounded quarterly? Construct a columnar table.

Deferred annuity. A deferred annuity is an annuity in which a number of periods are to expire before the periodic payments or rents are to begin.

The amount of a deferred annuity is the same as the amount of one which is not deferred, since no payments are made until after the time of deferment has expired.

The present value of a deferred annuity is the value at the beginning of the period of deferment.

There are two methods of computing the present value of a

deferred annuity. In each of these two distinct operations are necessary.

First method. Procedure: (*a*) Determine the present value of an ordinary annuity of 1 at the given rate and for the number of periods corresponding to the number of rents.

(*b*) Multiply the present value of 1 found in (*a*) by the number of dollars in each rent.

(*c*) Multiply the present value of the annuity found in (*b*) by the present value of 1 for the number of deferred periods.

Example

Find the present value of an annuity contract which calls for four equal annual payments of $100 each, the first rent to be paid at the end of the seventh year. Interest is to be calculated at 6%.

An analysis shows that this is an ordinary annuity for 4 years, deferred for 6 years.

Formula

Let: n = the number of rents
And: m = the number of deferred periods

$$Ra_{\overline{n}|i} \cdot v^m = \text{Present value of deferred annuity.}$$

Arithmetical Substitution

$$100\left[\frac{1-\dfrac{1}{(1.06)^4}}{.06}\right]\left[\frac{1}{(1.06)^6}\right] = \$244.28.$$

Short Solution, Part 1

1 ÷ 1.262477 = .7920937, present value of 1 due at the end of 4 years at 6%
1 − .7920937 = .2079063, compound discount on 1 due at the end of 4 years at 6%
.2079063 ÷ .06 = 3.465105, present value of annuity of 1 for 4 years at 6%, or $a_{\overline{4}|6\%}$ from Table 5, page 532
$100 × 3.465105 = $346.51, present value of annuity of $100 for 4 years

Short Solution, Part 2

$(1.06)^6$ = 1.418519
1 ÷ 1.418519 = .7049605, present value of 1 due at the end of 6 years at 6%
$346.51 × .704605 = $244.28, present value of deferred annuity

Verification, Part 1

Present time:	
Present value of deferred annuity	$244 28
Multiply by $(1.06)^6$	1 418519
End of deferred period:	
Value at end of deferred period	$346.51

End of 1st period:		
Rent	$100 00	
Interest on $346.51 at 6%	20 79	
Amortization		79.21
Balance		$267 30
End of 2nd period:		
Rent	$100 00	
Interest on $267.30 at 6%	16.04	
Amortization		83 96
Balance		$183.34
End of 3rd period:		
Rent	$100 00	
Interest on $183.34 at 6%	11 00	
Amortization		89 00
Balance		$ 94 34
End of 4th period:		
Rent	$100 00	
Interest on $94.34 at 6%	5 66	
Amortization		94 34

Second method. Procedure: (a) Compute the present value of an ordinary annuity of 1 at the given rate per cent, for a number of periods equal to the sum of the periods of the annuity and deferred periods, or $n + m$ periods.

(b) Compute the present value of an ordinary annuity of 1, at the given rate and for a number of periods equal to the deferred periods, or m periods.

(c) Find the difference between the present values found in (a) and (b).

(d) Multiply the difference found in (c) by a number equal to the number of dollars in each rent.

Formula

$$R(a_{\overline{n+m}|i} - a_{\overline{m}|i}) = \text{Present value of deferred annuity.}$$

Arithmetical Substitution

$$100\left[\left(\frac{1-\dfrac{1}{(1.06)^{10}}}{.06}\right)-\left(\frac{1-\dfrac{1}{(1.06)^{6}}}{.06}\right)\right] = \$244.28.$$

Solution

$(1.06)^{10} = 1.790847$, compound amount of 1 at 6% for 10 periods

$1 \div 1.790847 = .5583948$, present value of 1 at 6% for 10 periods

$1 - .5583948 = .4416052$, compound discount on 1 at 6% for 10 periods

$.4416052 \div .06 = 7.360087$, present value of annuity of 1 at 6% for 10 periods, or $a_{\overline{10}|6\%}$ from Table 5, page 532.

$(1.06)^6 = 1.418519$, compound amount of 1 at 6% for 6 periods

$1 \div 1.418519 = .7049605$, present value of 1 at 6% for 6 periods

$1 - .7049605 = .2950395$, compound discount on 1 at 6% for 6 periods

$.2950395 \div .06 = 4.917324$, present value of annuity of 1 at 6% for 6 periods, or $a_{\overline{6}|6\%}$ from Table 5, page 532.

$7.360087 - 4.917324 = 2.442763$, difference in present value of annuities of 1 for 10 periods and 6 periods

$\$100 \times 2.442763 = \$244.28$, present value of deferred annuity of \$100

Probably most of the difficulties in the solution of problems such as the above arise from failure to make a complete analysis before beginning the work. Problems of this type may be analyzed and solved in various ways. Be sure to see each problem in all its parts before attempting to calculate the amounts.

Problems

Find the present value of each of the following deferred annuities. (NOTE: If interest is to be compounded semiannually, quarterly, or monthly, this same condition usually prevails during the period of deferment.)

	Payments	*Payments Made*	*Rate*	*Number of Rents*	*Years Deferred*
1.	\$100	Annually	5%	5	5
2.	\$500	Annually	4%	6	4
3.	\$250	Semiannually	4%	10	5
4.	\$200	Quarterly	6%	16	3
5.	\$100	Quarterly	6%	12	5
6.	\$ 50	Monthly	6%	48	3

7. What is the present value of an annuity contract in which the A.B. Company agrees to pay to Mr. Ladd a monthly installment of \$40 for 10 years, the first payment to be made 10 years hence? Money is worth 6%, interest convertible monthly during the annuity period, and annually during the deferred period.

8. A company is issuing \$100,000 of 4%, 20-year bonds, which it wishes to pay at maturity by means of a sinking fund in which equal annual deposits are to be made. The board of directors wishes to assume that this fund will earn 5% interest for the first 10 years, and 4% for the last 10 years. What is the annual deposit required?

Given:

	5%	*4%*
S_{10}	12.578	12 006
$(1 + i)^{10}$	1 629	1.480

9. A lease has 5 years to run at \$1,200 a year, with an extension for a further 5 years at \$1,500 a year. The payments are due at the end of each year. If money is worth 5%, what should be the sum paid now in lieu of the 10 years' rent?

10.* On December 31, 1943, A is indebted to B in the following amounts:

\$1,500, due December 31, 1944, without interest
\$3,500, due December 31, 1946, with interest at the rate of 6% payable annually
\$5,000, due December 31, 1948, with interest at the rate of 6% from December 31, 1943, not payable until maturity of note but to be compounded annually
\$6,000, due December 31, 1949, with interest at the rate of 5% payable annually

On this date (December 31, 1943), A learns that on December 31, 1947, he will fall heir to \$200,000, and he arranges with B to cancel the four notes in exchange for one note due in 4 years. It is then agreed that the new note shall include interest to maturity calculated at 5%, compounded annually, and that B shall not lose by the exchange.

What will be the amount of the new note?

Given at 5%:

$$v^1 = .9523810 \qquad (1+i)^1 = 1.0500000$$
$$v^2 = .9070295 \qquad (1+i)^2 = 1.1025000$$
$$v^3 = .8638376 \qquad (1+i)^3 = 1.1576230$$
$$v^5 = .7835262 \qquad (1+i)^4 = 1.2155062$$

Given at 6%:

$$(1+i)^3 = 1.1910160$$
$$(1+i)^5 = 1.3382256$$

11.* On January 1, 1938, A leased a building to B for the period ending December 31, 1952, at an annual rental of \$7,000 payable annually in advance. Subject to this lease, A leased the same property to C on January 1, 1944, for a term of 50 years at an annual rental of \$10,000, payable annually in advance, C to receive the rental of \$7,000 payable by B during the remainder of B's lease. For this lease C paid to A an additional \$1,500 as a bonus.

Omitting all consideration of income tax questions, how should the various accounts appear on C's books if he calculates interest on the investment at 6% per annum?

Given at 6%:

$$(1+i)^9 = 1.689479 \qquad v^9 = .591899$$
$$(1+i)^{10} = 1.790848 \qquad v^{10} = .558395$$
$$(1+i)^{41} = 10.902861 \qquad v^{41} = .091719$$

12.† The Belgian pre-armistice debt to the United States amounted to \$171,800,000. The settlement provided that no interest was to be charged on this part of the war debt and that graduated payments on account of principal were to be made, totaling \$9,400,000, by June 15, 1931, the balance to be payable at the rate of \$2,900,000 per annum for 56 years.

Assuming an interest rate of 3% per annum, calculate the loss to the United States by the waiving of interest calculated at June 15, 1931.

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* Adapted from American Institute Examination.

† American Institute Examination.

The present value of $1.00 at 3% due in 56 years is $0.1910361, and in 56 years $1.00 at 3% will amount to $5.2346131.

13. A note, the face value of which is $1,000, bears interest at the rate of 8% per annum, and is payable in monthly installments of $25, including interest. It is desired to discount this note at the bank, so that the bank shall have an effective rate of 12% per annum. What is the amount of the discount to be deducted by the bank?

Perpetuity. A perpetuity is defined as a series of periodic payments which are to run indefinitely. This form of annual payment is most often found as the effect of the establishment of an endowment fund, the rents of which are to be used for a special purpose. As the endowment fund is never to be returned, the amount has no meaning, but its present value is the value of the periodic payment divided by the prevailing rate of interest. The present value of a perpetuity is denoted by a_∞.

Procedure: Divide the periodic rent by the current rate per cent expressed decimally, to find the present value of the perpetuity.

Example 1

It is desired to establish a fund the annual income from which, at 6%, will be $300. Find the present value of the perpetuity.

Formula	*Arithmetical Substitution*
$\frac{\text{Rent}}{i} = A_\infty$	$\frac{\$300}{.06} = \$5,000$

Solution

$300 ÷ .06 = $5,000, the amount of the endowment fund

Example 2

Find the present value of a perpetuity of $50 a month at 6% nominal convertible monthly.

Solution

$$\frac{\$50}{.005} = \$10,000$$

Example 3

What is the present value of a perpetuity of $500 a year, if money is worth 4% convertible semiannually?

Solution

$$\frac{\$500}{.0404} = \$12,376.24$$

Perpetuities payable at intervals longer than a year. In this case the annual interest on the principal will accumulate during the interval to equal the payment due at the end of that time.

Example

What is the present value of a perpetuity of \$1,000 every 5 years, if interest is at 4% a year?

Solution

$$\frac{\$1{,}000}{.04} \cdot \frac{1}{s_{\overline{5}|4\%}} = \$25{,}000 \times 0.1846271 = \$4{,}615.68$$

Problems

1. If a farm produces a net annual income of \$3,600, what is its present value if money is worth 5%?

2. What is the present value of a perpetuity of \$3,500 payable every 5 years, if money is worth 4% convertible semiannually?

3. Find the present value of a perpetuity of \$250 a year if money is worth 6%.

4. What is the present value of a perpetuity of \$600 a year if money is worth 4% convertible quarterly?

5. Find the annual rent of a perpetuity whose present value is \$15,000, if interest is 5% a year.

6. Find the present value of a perpetuity of \$10,000, payable every 5 years, if money is worth $4\frac{1}{2}$% a year.

Review Problems

1. Brown sets aside \$500 at the beginning of each year to provide for his daughter's college education. If he invests the money at 3% effective, what will be the amount following the 10th payment?

2. Find the present value of a premium of \$27.42, assuming that the insured will live to pay 20 premiums, money being worth 4% effective.

3. White bought a house, agreeing to pay \$1,000 down and \$500 each six months until he paid \$8,000. If money is worth 5% effective, what should be the cash price of the house?

4. Find the present value of an annuity of \$600 a year for 10 years, if money is worth 4% effective: (*a*) deferred 5 years, (*b*) deferred 8 years.

5. If you deposit \$150 in a savings bank at the beginning of each quarter and the bank pays 3% nominal convertible quarterly, how much will you have to your credit at the end of 5 years?

6. Cole, at age 22, takes a 20-payment life insurance policy of \$1,000 on which the premium is \$27.42 payable at the beginning of each year. If he should die at the end of 10 years just before the eleventh premium is due, by how much would his estate be increased by having taken the insurance instead of having put the premiums into a savings bank paying 2% effective?

7. Find the present value of an annuity of \$1,000 a year to be paid in quarterly installments for 12 years, deferred for 5 years, if money is worth 3% effective.

8. An insurance premium of \$64.50 is payable semiannually in advance for 20 years. If interest is at 3% convertible semiannually, find the amount of the payments at the end of 20 years.

9. Attached to a $100 bond are coupons worth $2.50 each six months for the next 20 years. What is the present value of these coupons, assuming money to be worth 4% effective?

10. How long will it take to accumulate $10,000 by investing $500 each six months at 3% convertible semiannually?

11. What rate of interest compounded monthly is equivalent to 6% compounded quarterly?

12. Find the annual payment required to accumulate $3,000 in 10 years, when money is worth 3% convertible semiannually.

13. If it takes an orchard 6 years to reach profitable maturity, and for a period of 20 years it is expected to yield a net income of $3,500 a year, what is the cash value of the orchard if money is worth 4% effective?

14. Find the present value of an annuity due of $250 a year payable annually for 6 years, if money is worth 5%.

15. Find the present value of an annuity of $75 a month for 8 years, 6 months, if money is worth 6% nominal converted monthly.

CHAPTER 33

Bond and Bond Interest Valuation

Definitions. A bond is a promise to pay a specified sum of money at a determinable future date. It differs from a note, in general, in that it is usually a long term obligation. Bonds are generally issued by a city, state, nation, or corporation, and seldom by individuals. The sum written in the body of the instrument is known as the face, par, or nominal par, and is the amount on which interest is calculated. Bonds usually have a par of $100, $500, or $1,000.

Interest payments may be made annually, semiannually, or quarterly. These interest payments are known as nominal interest, or cash interest.

The rate per cent earned on the actual money invested is termed the effective or investment interest. The cash rate and the effective rate are not the same unless the bond is purchased at par. Quotations are sometimes made "on a basis," which means at an effective rate on the money invested. The price of a bond will usually be either above or below par; because the cash rate is either above or below the effective rate.

A bond is said to be "redeemed" when it is bought back by the company which issued it. If the market price of a bond is more than the par value, the bond is said to be above par; or if less, the bond is said to be below par. Bonds sold above par are said to be sold "at a premium," and if sold below par, they are said to be sold "at a discount."

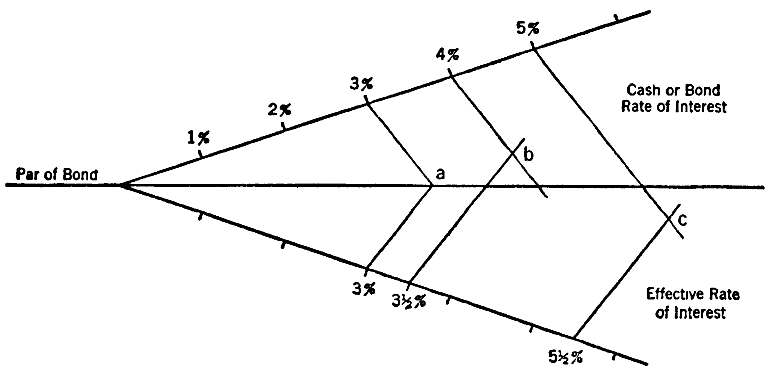
Bonds are usually sold on the open market for whatever they will bring. Whether they bring more or less than par depends upon:

(1) The cash or coupon rate of interest.
(2) The redemption price.
(3) The current rate of interest.
(4) The length of time until maturity.
(5) The character of the security.

Only the first four of the above can be mathematically considered.

Bonds sold at par. It is apparent that if a man desires to buy a certain bond which has a face value of $1,000, and this bond bears a cash rate of interest exactly the same as that which his money is worth on the market for other securities of the same general class, he will be willing to pay $1,000 for the bond. In such a case, the interest that he will receive from this bond will be equivalent to the interest that he would receive from any other investment of the same general class. But if he considered the purchase of another bond of $1,000, which had a cash rate of interest lower than that of other investments of the same general class, it is apparent that he would not pay a full $1,000 for this bond. And again, if there were on the market securities of the same general class paying a higher rate per thousand, he would be willing to pay more than the par value for a bond of this class.

This may be illustrated graphically, as follows:



1. Cash or bond rate of interest has a tendency to lift the price of the bond above par.

2. Effective rate of interest has a tendency to bring the price of the bond below par.

a. This is an illustration of an equal pull of cash and effective interest. If all other factors were equal, the result would be a price at par.

b. If the cash rate is larger than the effective rate, the price should be above par.

c. If the effective rate is larger than the cash rate, the pull is below par.

Bonds purchased at a discount or at a premium. If a bond is purchased at a discount, the effective rate is higher than the cash rate, for two reasons: (*a*) the investment is less than par; and (*b*) the investor makes additional income equal to the difference between the cost and the par to be collected at maturity. On the other hand, if a bond is purchased at a premium, the effective rate is lower than the cash rate, for two reasons: (*a*) the investment is

more than par; and (*b*) the investor loses the difference between the price paid and the par to be collected at maturity.

Price and rate of yield. In business, the usual questions which arise with regard to bonds are: "What price should be paid for bonds if a certain investment rate of interest is desired by the investor?" and "What rate will a bond, purchased at a certain price, produce?"

Use of bond tables. In the calculation of the cost, bond tables are generally used. They are reproduced below in order that the student may familiarize himself with their use.

Bond table, first form. The following is a very simple form of bond table, in which only values, effective rates, and nominal rates are shown:

TABLE OF BOND VALUES

FIVE YEARS, INTEREST PAYABLE SEMIANNUALLY

Effective Rate	*Nominal Rate*						
	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	6	7
$5\frac{1}{2}$	89 20	91 36	93 52	95 68	97 84	102.16	106.48
$5\frac{5}{8}$.	88.70	90 85	93 00	95.16	97 31	101.61	105 92
$5\frac{3}{4}$	88 20	90 34	92 49	94.63	96 78	101.07	105 37
$5\frac{7}{8}$.	87.70	89 84	91 98	94.12	96 26	100.53	104 81
6	87 20	89 34	91.47	93.60	95 73	100.00	104 27

Example

What is the purchase price of a 5-year, 5% bond (interest payabie semiannually), bought on a 6% basis?

Referring to the table, the answer is found to be $95 73.

Problems

1. A $1,000 bond due in 5 years bears interest at 4%, payable semiannually, and is bought on a $5\frac{1}{2}$% basis. What is the purchase price, or value?

2. A $1,000 bond due in 5 years bears interest at 6%, payable semiannually, and is purchased on a $5\frac{1}{2}$% basis. State the value of the bond.

3. What price can be paid for city bonds of $1,000 each, due in 5 years, interest at 5%, payable semiannually, bought on a 6% basis?

4. If John Jones bought one $4\frac{1}{2}$%, $1,000 bond for $946.30, what per cent may he expect on his investment?

5. The coupon rate is $4\frac{1}{2}$%; the number of bonds is 5; the purchase price is $4,680.00. What per cent is made on the investment?

Bond table, second form. The most common form of bond table shows the "Price" as indexed between the "Nominal" and the "Effective" rates. A new and very efficient form of bond

table is that used by Johnson, Stone, Cross, and Kircher in their volume, *Yields of Bonds and Stocks* (New York, Prentice-Hall, Inc., enlarged edition, 1938). In this form of table the "Effective" rate is indexed between the "Nominal" rate and the "Price." An additional feature is the showing of the cash rate of return. A page from *Yields of Bonds and Stocks* is reproduced on page 371.

Example

What will be the rate of yield of a bond purchased at $97.25, if it bears 5% interest, payable semiannually, and matures in 5 years?

Procedure: (*a*) Turn to the 5% table.

(*b*) Find the column headed 5 years.

(*c*) In the column at the left, find the price $97.25.

(*d*) Find the rate at the intersection of the year column and the price line. In this case it is 5.639%.

Interpolating in bond tables. It frequently happens that a given yield or a given price is not listed in a bond table. In such cases, however, the desired price or yield rate can be obtained by interpolation.

Example

What price can an investor pay for a bond due in 5 years, if he wishes to obtain a yield of 6% and the bond bears interest at 5%, payable semiannually?

Procedure: (*a*) Turn to the 5% table.

(*b*) Find the column headed 5 years.

(*c*) Select from this column two effective rates, one just above and the other just below the rate desired.

(*d*) In the price column at the left, select the prices corresponding to the rates selected in (*c*).

(*e*) The rate of yield is obtained by interpolating between the yields of the prices found in (*c*).

Solution

When yield = 6.057, the price is $95.50
" " = 6.000, " " " ______
" " = 5.996, " " " 95.75

$$6.057 - 5.996 = 0.061$$
$$6.057 - 6.000 = 0.057$$
$$\$95.75 - \$95.50 = \$.25$$

Since a difference of 0.061 in the rate of yield means a difference of 25¢ in the price, a difference of 0.057 in the rate of yield will mean a difference of $\frac{57}{61}$ of .25; hence the price sought is $\frac{57}{61} \times .25$ greater than $95.50. That is:

$$\tfrac{57}{61} \times .25 = .2336$$
$$\$95.50 + .2336 = \$95.7336 \text{ (Answer)}$$

Example

What is the rate of yield on a 5% bond due in 4 years, interest payable semiannually, if the bond is purchased at $96\frac{7}{8}$?

TABLE OF YIELDS OF 5% BOND

YIELDS IN PER CENT PER ANNUM, CORRECT TO THE NEAREST FIVE TEN-THOUSANDTHS OF 1%, INTEREST PAYABLE SEMIANNUALLY

Price	3 Years	3½ Years	4 Years	4½ Years	5 Years	5½ Years	6 Years	Current Income
94	7.262	6.961	6.736	6.561	6.422	6.308	6.213	5.319
94¼	7.164	6.876	6.661	6.494	6.361	6.252	6.161	5.305
94½	7.067	6.792	6.586	6.427	6.299	6.195	6.109	5.291
94¾	6.970	6.708	6.512	6.360	6.238	6.139	6.057	5.277
95	6.873	6.624	6.438	6.293	6.178	6.083	6.005	5.263
95¼	6.776	6.540	6.364	6.227	6.117	6.028	5.953	5.249
95½	6.680	6.457	6.290	6.160	6.057	5.972	5.902	5.236
95¾	6.584	6.374	6.216	6.094	5.996	5.917	5.850	5.222
96	6.489	6.291	6.143	6.028	5.936	5.861	5.799	5.208
96¼	6.394	6.209	6.070	5.962	5.876	5.806	5.748	5.195
96½	6.299	6.126	5.997	5.897	5.817	5.751	5.697	5.181
96¾	6.204	6.044	5.924	5.832	5.757	5.697	5.646	5.168
97	6.110	5.962	5.852	5.766	5.698	5.642	5.596	5.155
97¼	6.016	5.881	5.780	5.701	5.639	5.588	5.545	5.141
97½	5.922	5.800	5.708	5.637	5.580	5.533	5.495	5.128
97¾	5.828	5.718	5.636	5.572	5.521	5.479	5.445	5.115
98	5.735	5.638	5.565	5.508	5.463	5.425	5.395	5.102
98¼	5.642	5.557	5.493	5.444	5.404	5.372	5.345	5.089
98½	5.550	5.477	5.422	5.380	5.346	5.318	5.295	5.076
98¾	5.457	5.397	5.351	5.316	5.288	5.265	5.246	5.063
99	5.365	5.317	5.281	5.252	5.230	5.211	5.196	5.051
99¼	5.274	5.237	5.210	5.189	5.172	5.158	5.147	5.038
99½	5.182	5.158	5.140	5.126	5.115	5.105	5.098	5.025
99¾	5.091	5.079	5.070	5.063	5.057	5.053	5.049	5.013
100	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000
100¼	4.909	4.921	4.930	4.937	4.943	4.948	4.951	4.988
100½	4.819	4.843	4.861	4.875	4.886	4.895	4.903	4.975
100¾	4.729	4.765	4.792	4.813	4.829	4.843	4.854	4.963
101	4.639	4.687	4.723	4.751	4.773	4.791	4.806	4.950
101¼	4.550	4.609	4.654	4.689	4.716	4.739	4.758	4.938
101½	4.460	4.532	4.585	4.627	4.660	4.687	4.710	4.926
101¾	4.371	4.454	4.517	4.565	4.604	4.636	4.662	4.914
102	4.282	4.377	4.449	4.504	4.548	4.584	4.615	4.902
102¼	4.194	4.301	4.381	4.443	4.493	4.533	4.567	4.890
102½	4.106	4.224	4.313	4.382	4.437	4.482	4.520	4.878
102¾	4.018	4.148	4.245	4.321	4.382	4.431	4.472	4.866
103	3.930	4.072	4.178	4.260	4.326	4.380	4.425	4.854
103¼	3.843	3.996	4.111	4.200	4.271	4.330	4.378	4.843
103½	3.755	3.920	4.044	4.140	4.216	4.279	4.331	4.831
103¾	3.669	3.845	3.977	4.079	4.162	4.229	4.285	4.819

* From Johnson, Stone, Cross, and Kircher, *Yields of Bonds and Stocks.* New York: Prentice-Hall, Inc., enlarged edition, 1938.

Solution

When price = $96.75,	the yield is	5.924
" " = 96.875,	" " "	....
" " = 97.000,	" " "	5.852

$97.00 − $96.75 = $.25
$97.00 − 96.875 = $.125
5.924 − 5.852 = 0.072

Since a difference of 25¢ in the price means a difference of 0.072 in the rate of yield, a difference of $12\frac{1}{2}$¢ in the price will mean a difference in the rate of yield of $\frac{12\frac{1}{2}}{25}$ of 0.072; hence the sought for rate of yield is $12\frac{1}{2}/25 \times 0.072$ less than 5.924. That is:

$$\text{Yield sought} = 5.924 - \left(\frac{12\frac{1}{2}}{25} \times 0.072\right)$$
$$= 5.924 - 0.036$$
$$= 5.888\% \text{ (Answer)}$$

Problems

By means of the foregoing bond table, supply the missing factor in each of the following:

	Par of Bonds	*Price*	*Nominal Rate*	*Effective Rate*	*Years*	*Interest Payable*
1.	$ 1,000	$ 977 50	5%	....	3	Semiannually
2.	1,000	....	5%	6%	4	"
3.	5,000	5,087.50	5%	....	$3\frac{1}{2}$	"
4.	4,000	....	5%	$4\frac{1}{2}$%	5	"
5.	10,000	9,812.50	5%	....	$5\frac{1}{2}$	"
6.	2,000	....	5%	$5\frac{1}{2}$%	6	"

Bond values computed without tables. If no bond tables are available, the values of bonds sold at a discount or at a premium may be found by either of two methods of calculation.

Bonds sold at a discount. The discount on bonds is the difference between the cost (which is below par) and the par value.

Let C = the par value of the bond;
C' = the redemption price when more than par;
r = the rate named in the bond, that is, the coupon or cash interest;
i = the rate of yield expected;
n = the number of years to maturity;
P = the cost of the bond.

Two methods of calculation are given.

First method. Procedure: (*a*) Compute the present value of the par of the bond at the investment rate of interest and for the number of periods between the date of purchase and the maturity of the bond, Cv^n.

(*b*) Compute at the effective interest rate the present value of an annuity for a number of periods equal to the number of bond-interest or cash-interest payments, the rents of the annuity to be of the same amount as the periodic bond-interest or cash-interest payments, $rCa_{\overline{n}|i}$.

(*c*) Add the present value of the par of the bond found in (*a*), and the present value of the annuity found in (*b*). The sum will be the present value of the bond.

Example

What will be the cost of a 5% bond for \$100, maturing in 4 years, bought so as to produce 6% effective interest, payable semiannually?

Formula

$$Cv^n + rCa_{\overline{n}|i} = P$$

Arithmetical Substitution

$$100\left(\frac{1}{(1.03)^8}\right) + 2.50\left(\frac{1 - \dfrac{1}{(1.03)^8}}{.03}\right) = \$96.49.$$

Solution, Part 1

Finding the present value of the par of the bond:

$(1.03)^8 = 1.2667701$, compound amount of 1 at 3% for 8 periods
$1 \div 1.2667701 = .7894092$, present value of 1 at 3% for 8 periods
$\$100 \times .7894092 = \$78.94$, present value of \$100 at 3% for 8 periods

Solution, Part 2

Finding the present value of an annuity the rents of which, \$2.50, are the same as the periodic interest payments on the bond:

$.7894092 =$ present value of 1, as found above
$1 - .7894092 = .2105908$, compound discount on 1 at 3% for 8 periods
$.2105908 \div .03 = 7.01969$, present value of an annuity of 1
$\$2.50 \times 7.01969 = \$17.549$, present value of an annuity of \$2.50

Solution, Part 3

Adding:

$\$78.94 + \$17.55 = \$96.49$, cost of bond

A convenient and condensed statement showing, for each period, the carrying value, the accumulation of discount, the coupon interest, and the effective interest may be made as follows.

End of Period	Effective Interest	Coupon Interest	Accumulation of Discounts	Carrying Value
				$ 96 49
1	$2 89	$2 50	$ 39	96 88
2	2 91	2 50	.41	97 29
3	2 92	2 50	.42	97 71
4	2 93	2 50	.43	98 14
5	2 94	2 50	.44	98 58
6	2 96	2 50	.46	99 04
7	2 97	2 50	47	99 51
8	2 99	2 50	.49	100 00

Second method. The theory of this method is that the $2.50 interest received will offset $2.50 of the $3.00 expected; therefore, the cost of the bond will be the par value of the bond less the present value at the effective interest rate of an annuity of $.50.

Procedure: (*a*) Compute the present value of an annuity of 1 at the effective interest rate for a number of periods equal to the number of periods that the bond has yet to run, $a_{\overline{n}|i}$.

(*b*) Calculate the difference between the number of dollars of income per period at the desired rate and at the cash rate, using as the basis in both cases the par of the bond, $Ci - Cr$.

(*c*) Multiply the present value of the annuity found in (*a*), by the difference found in (*b*); the result will be the discount on the par value of the bond, $(Ci - Cr) \cdot a_{\overline{n}|i}$.

(*d*) Deduct the discount found in (*c*) from the par value of the bond, and the result will be the price.

Formula

$$C - [(Ci - Cr) \cdot a_{\overline{n}|i}] = P$$

Arithmetical Substitution

$$100 - \left[(3.00 - 2.50)\left(\frac{1 - \dfrac{1}{(1.03)^8}}{.03}\right)\right] = \$96.49$$

Solution, Part 1

$(1.03)^8 = 1.2667701$, compound amount of 1 at 3% for 8 periods

$1 \div 1.2667701 = .7894092$, present value of 1 at 3% for 8 periods

$1 - .7894092 = .2105908$, compound discount on 1 at 3% for 8 periods

$\$.2105908 \div .03 = \$7.01969$, present value of an annuity of 1

Solution, Part 2

$3.00 - 2.50 = .50$, difference between effective and cash interest for 1 period

$\$7.01969 \times .50 = \$3.51$, discount on bond

Solution, Part 3

$100 − $3.51 = $96.49, cost of bond

Verification

First Period:		
Cost of bond		$ 96.49
Interest at 3% on $96.49	$2.89	
Coupon interest	2.50	
Accumulation of discount		39
		$ 96.88
Second Period:		
Interest at 3% on $96.88	$2.91	
Coupon interest	2.50	
Accumulation of discount		41
		$ 97.29
Third Period:		
Interest at 3% on $97.29	$2.92	
Coupon interest	2.50	
Accumulation of discount		42
		$ 97.71
Fourth Period:		
Interest at 3% on $97.71	$2.93	
Coupon interest	2.50	
Accumulation of discount		43
		$ 98.14
Fifth Period:		
Interest at 3% on $98.14	$2.94	
Coupon interest	2.50	
Accumulation of discount		.44
		$ 98.58
Sixth Period:		
Interest at 3% on $98.58	$2.96	
Coupon interest	2.50	
Accumulation of discount		46
		$ 99.04
Seventh Period:		
Interest at 3% on $99.04	$2.97	
Coupon interest	2.50	
Accumulation of discount		.47
		$ 99.51
Eighth Period:		
Interest at 3% on $99.51	$2.99	
Coupon interest	2.50	
Accumulation of discount		.49
Par of bond		$100.00

Problems

1. What should be the purchase price of a $1,000, 5-year, 5% bond (interest payable semiannually), bought so that it will produce $6\frac{1}{4}$%? Prove your work by means of the table.

2. If money is worth 6%, interest payable semiannually, what should be the purchase price of five $100 bonds, bearing a cash rate of 5%, and having 5 years to run? Prove your answer by means of the table.

3. Construct in columnar form a table showing the carrying value, the cash interest, the effective interest, and the amortization of a 5-year, 6% bond of $500, bought on a 7% basis (interest payable semiannually).

4. A $500 bond, maturing in 6 years and bearing interest at 6%, payable semiannually, is bought on an 8% basis. Construct a columnar table, as in problem 3.

5. Show in columnar form the carrying value, the cash interest, the effective interest, and the accumulation of discount for a 4-year bond, the par value of which is $1,000. The bond bears 5% interest, payable semiannually; the effective rate is 6%, convertible semiannually.

Bonds sold at a premium. As stated previously, if the effective rate is less than the coupon rate, the bond will sell at a premium, which means that it will be priced above par. When bonds sell at a premium, part of the money received for each coupon is used to cancel part of the premium paid for the bond.

As in the case of discount on bonds, two methods of calculating the price of a bond sold at a premium are in common use.

First method. Procedure: The procedure here is the same as the procedure for the first method of finding the price of a bond purchased at a discount.

Example

What price should be paid for a $100, 6%, 4-year bond, in order that the investor may realize 5% on his investment? Interest coupons are payable semiannually.

Formula

$$Cv^n + Cr \cdot a_{\overline{n}|i} = P$$

Arithmetical Substitution

$$100\left(\frac{1}{(1.025)^8}\right) + 3\left(\frac{1 - \frac{1}{(1.025)^8}}{.025}\right) = \$103.58.$$

Solution, Part 1

Finding the present value of the face of the bond:

$(1.025)^8 = 1.2184029$, compound amount of 1 at $2\frac{1}{2}\%$ for 8 periods

$1 \div 1.2184029 = .8207466$, present value of 1 at $2\frac{1}{2}\%$ for 8 periods

$\$100 \times .8207466 = \$82.07$, present value of face of bond

Solution, Part 2

Finding the present value of the annuity of $3:

$1 - .8207466 = .1792534$, compound discount on 1 at $2\frac{1}{2}\%$ for 8 periods

.1792534 ÷ .025 = 7.1701372, present value of an annuity of 1 at $2\frac{1}{2}\%$ for 8 periods

\$3 × 7.1701372 = \$21.51, present value of an annuity of \$3

Solution, Part 3

Adding:

\$82.07 + \$21.51 = \$103.58, cost of bond

COLUMNAR TABLE SHOWING VERIFICATION

End of Period	*Effective Interest*	*Coupon Interest*	*Amortization*	*Carrying Value*
				\$103 58
1	\$2 59	\$3 00	\$ 41	103 17
2	2 58	3 00	42	102 75
3	2 57	3 00	43	102 32
4	2 56	3 00	44	101 88
5	2 55	3 00	45	101.43
6	2 54	3 00	46	100 97
7	2 52	3 00	.48	100.49
8	2 51	3 00	49	100 00

Second method. Procedure: (*a*) Compute the present value of an annuity of 1 at the investment rate of interest and for a number of periods equal to the number of unexpired periods of the bond, $a_{\overline{n}|i}$.

(*b*) From the number of dollars of one cash interest subtract the number of dollars found by multiplying the par of the bond by the effective rate of interest, $Cr - Ci$.

(*c*) Multiply the present value of the annuity found in (*a*) by the number of dollars of the excess of the investment interest found in (*b*). The result obtained is the premium on the bond, $(Cr - Ci) \cdot a_{\overline{n}|i}$.

(*d*) Add the par value of the bond and the premium found in (*c*) to obtain the present value of the bond at a premium.

Formula

$$[(Cr - Ci) \cdot a_{\overline{n}|i}] + C = P$$

Arithmetical Substitution

$$\left[(3\ 00 - 2.50)\left(\frac{1 - \frac{1}{(1.025)^8}}{.025}\right)\right] + 100 = \$103.58$$

Solution, Part 1

$(1.025)^8$ = 1.2184029, compound amount of 1 at $2\frac{1}{2}\%$ for 8 periods

1 ÷ 1.2184029 = .8207466, present value of 1 at $2\frac{1}{2}\%$ for 8 periods

1 − .8207466 = .1792534, compound discount on 1 at $2\frac{1}{2}\%$ for 8 periods

.1792534 ÷ .025 = 7.1701372, present value of an annuity of 1 at $2\frac{1}{2}\%$ for 8 periods
\$.50 × 7.1701372 = \$3.58, present value of an annuity of \$.50

Solution, Part 2

Adding: \$100 + \$3.58 = \$103.58

Verification

First Period:		
Cost of bond		\$103.58
Coupon interest	\$3 00	
Less $2\frac{1}{2}\%$ interest on \$103.58	2 59	
Amortization of premium		.41
		\$103.17
Second Period:		
Coupon interest	\$3 00	
Less $2\frac{1}{2}\%$ interest on \$103.17	2 58	
Amortization of premium		.42
		\$102.75
Third Period:		
Coupon interest	\$3 00	
Less $2\frac{1}{2}\%$ interest on \$102.75	2 57	
Amortization of premium		43
		\$102.32
Fourth Period:		
Coupon interest	\$3 00	
Less $2\frac{1}{2}\%$ interest on \$102.32	2 56	
Amortization of premium		.44
		\$101.88
Fifth Period:		
Coupon interest	\$3 00	
Less $2\frac{1}{2}\%$ interest on \$101.88	2 55	
Amortization of premium		45
		\$101 43
Sixth Period:		
Coupon interest	\$3 00	
Less $2\frac{1}{2}\%$ interest on \$101.43	2 54	
Amortization of premium		46
		\$100.97
Seventh Period:		
Coupon interest	\$3 00	
Less $2\frac{1}{2}\%$ interest on \$100.97	2 52	
Amortization of premium		.48
		\$100 49
Eighth Period:		
Coupon interest	\$3 00	
Less $2\frac{1}{2}\%$ interest on \$100.49	2.51	
Amortization of premium		.49
Par of bond		\$100.00

Problems

Fill in the price in each of the following; the interest is payable semiannually:

	Face of Bond	*Time to Run*	*Cash Interest*	*Effective Interest*	*Price*
1.	$ 100	5 years	6%	7%	$...........
2.	1,000	10 years	5%	6%	....
3.	5,000	15 years	$5\frac{1}{2}$%	6%	....
4.	2,000	12 years	6%	$5\frac{1}{2}$%	....
5.	3,000	6 years	$6\frac{1}{2}$%	$4\frac{1}{2}$%	....
6.	5,000	20 years	$5\frac{1}{2}$%	5%	....

In problems 7, 8, 9, and 10, set up a columnar table showing: (*a*) the number of periods; (*b*) the effective interest; (*c*) the coupon interest; (*d*) the amortization of premium or discount; and (*e*) the carrying value. The interest is payable semiannually.

	Face of Bond	*Time to Run, Years*	*Cash or Coupon Interest*	*Effective Interest*	*Price*
7.	$ 100	4	5%	6%	$...........
8.	3,000	$3\frac{1}{2}$	6%	5%	....
9.	7,500	4	$6\frac{1}{2}$%	$5\frac{1}{2}$%	....
10.	5,000	5	4%	5%	....

11.* A $10,000, 5% coupon bond is bought on a 4% basis. It is due $1\frac{1}{2}$ years hence, and interest is payable semiannually. Find the cost of the bond.

12. What is the difference in the purchase price of two $1,000, 20-year bonds, bought to yield 6%, if one of the bonds has a semiannual coupon of $25, while the other has a semiannual coupon of $35?

13. Davis died on April 1, 1933. His estate contained five $1,000 bonds of the X.Y.Z. Company, bearing 6% interest, payable July 1st and January 1st. The bonds were due on July 1, 1938, and were inventoried at $104\frac{1}{2}$. On July 1, 1933, the trustee purchased five more of the same bonds on a 5% basis. Compute the price paid by the trustee for the bonds. Assume the value of $1, due after ten periods at $2\frac{1}{2}$%, to be $.781198402.

14. Find the price for a $1,000 bond bearing interest at $5\frac{1}{2}$%, payable May 1 and November 1, maturing May 1, 1953, if bought on May 1, 1942 at a price to yield the purchaser 5%.

15. Suppose the bond described in Problem 14 were bought to yield the purchaser 6%. Find the price.

Values of bonds between interest dates. Heretofore, in finding the values of bonds we have used even periods in the calculation of the interest. However, if it is desired to find the value of a bond at a date other than an interest date, additional computations are necessary. Two factors must be taken into account: (1) accrued interest at the cash rate must be computed for the fraction of an

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* C. P. A. Examination.

interest period elapsed; and (2) the amortization of the premium or the accumulation of the discount must be computed for the fraction of an interest period elapsed.

Interest accrued between interest dates. Finding the amount of the accrued interest for a fractional part of a period is a simple computation which requires no explanation. The accrued interest must always be considered when the exact value of an investment is being determined.

Bond discount or premium between interest dates. For practical purposes, the amortization of the premium or the accumulation of the discount between interest dates may be calculated on a proportional basis, by means of interpolation. This method gives a fair degree of accuracy. The amount may be readily found if bond tables are used.

Illustration of the practical process of calculating the value of a bond bought at a discount.

Procedure: (*a*) By the use of bond tables, or of formulas previously given, determine the value of the bond at the interest date just preceding the purchase date, and the value of the bond at the interest date just subsequent to the purchase date.

(*b*) Calculate the discount for one period by finding the difference between the values determined in (*a*).

(*c*) Calculate the part of the discount which is in the same proportion to the discount for one period, found in (*b*), as the fractional part of the period which has elapsed is to the total period.

(*d*) Add the discount for the fractional period, found in (*c*), to the value of the bond at the interest date just preceding the purchase date.

(*e*) Determine the accrued interest, at the rate specified in the bond, on the par of the bond for the time expired since the last interest date.

(*f*) Add the accrued interest found in (*e*) to the amount found in (*d*); the sum is the value of the bond, with interest.

Example

On March 1, 1944, what was the value of a $1,000, $4\frac{1}{2}\%$ bond, due January 1, 1949? Interest coupons are payable January 1 and July 1, and money is worth 6%, interest compounded semiannually.

Solution

Value 9 periods before maturity, at 6%	$941 60
Value 10 periods before maturity, at 6%	936 02
Accumulation of discount during 1 period	$ 5 58

Interpolation

Accumulation of discount during 1 period..............	$ 5.58
As two months of the six months' period have elapsed, the simple proportional part of the discount accumulated is $\frac{1}{3}$ of $5.58, or $1.86.	
Value 10 periods before maturity. . .	936 02
Add 2 months' accumulation of discount	1 86
Value 9 periods and 4 months before maturity ..	$937.88
Add accrued portion of next interest coupons	7 50
Total value of bond, with interest............. .. .	$945 38

Theoretical procedure illustrated. As the discount accumulates at the rate of 3% for one whole period, and as exactly one third of a period has expired, the accumulation may be expressed by a fraction:

$$\frac{(1.03)^{\frac{1}{3}} - 1}{(1.03) - 1} \times 5.58.$$

When the 3rd root of (1.03) is extracted, the fraction becomes:

$$\frac{(1.009902) - 1}{(1.03) - 1} \times 5.58.$$

Simplifying, $\frac{.009902}{.030000} \times 5.58$	$1 842
Accumulation of discount by interpolation (as above).	1.86
Accumulation of discount by theoretical process	1 842
Error by interpolation in the practical process of calculation	$ 018

Illustration of the practical process of calculating the value of a bond bought at a premium.

Procedure: (*a*) Determine from bond tables the value of the bond at the interest date just preceding the purchase date, and the value of the bond at the first interest date subsequent to the purchase date.

(*b*) Determine the premium for one period by finding the difference between the values found in (*a*).

(*c*) Calculate the part of the premium which is in the same proportion to the premium for one period as the expired part of the period is to the whole period.

(*d*) Deduct the premium found in (*c*) from the value of the bond at the interest date just preceding the purchase date.

(*e*) Determine the accrued interest on the par of the bond for the expired fractional part of the period.

(*f*) Add the accrued interest found in (*e*) to the amount found in (*d*); the sum is the purchase price of the bond, with accrued interest.

Example

On May 1, 1943, what should have been paid for a $1,000, 6% bond, due January 1, 1947, if interest coupons were payable January 1 and July 1, and money was worth 5%, interest compounded semiannually?

Solution

Value 8 periods before maturity, at 5%	$1,035 85
Value 7 periods before maturity, at 5%	1,031 75
Amortization of premium during 1 period . .	$ 4 10

Interpolation

Amortization of premium during 1 period	$ 4 10	
As 4 months of the 8th period have expired, the proportional part of the 6 months' period is two-thirds. Therefore, the amortization which has taken place is $\frac{2}{3}$ of $4.10, or $2.73.		
Value 8 periods before maturity		$1,035 85
Deduct 4 months' amortization of premium		2 73
Value 7 periods and 2 months before maturity		$1,033 12
Interest at 6% for 1 period.	$30 00	
As 4 months have expired since any interest was paid, there is due $\frac{4}{6}$, or $\frac{2}{3}$, of $30 .		20 00
Value of bond, with accrued interest		$1,053 12

Bonds bought on a yield basis. For bonds bought on a strictly yield basis, the following procedure may be used:

To the price of the bond at the last preceding interest date, add interest thereon at the effective (yield) rate for the expired portion of the period during which the purchase is made.

Example

A $1,000 bond maturing October 1, 1952 with interest at 6% payable April 1 and October 1 was bought on July 1, 1942 at a price to yield the investor 5%. What price was paid for the bond?

Solution

(*a*) To obtain the price of the bond on April 1, 1942, which was the last interest payment date prior to the date of purchase, follow the procedure on page 376. The price of the bond on April 1, 1942 is found to be $1,080.92.

(*b*) To determine the price of the bond on July 1, 1942, compute the elapsed time from April 1 to July 1, which is 3 months or one-half a period. Then,

$$\$1{,}080.92 \times (1.025)^{\frac{1}{2}} = \$1{,}094.35$$

Problems

1. A $100 bond bears 5% interest, payable semiannually, and is due in 5 years and 4 months. What price, plus accrued interest, should an investor pay for the bond if he wishes his investment to produce 4%?

2. A \$1,000, 5% bond is due in 6 years and 3 months, and interest is payable semiannually. The effective interest rate is 6%. What is the value of the bond, with accrued interest?

3. A \$1,000, $5\frac{1}{2}$% bond, with interest payable annually, is redeemable at par in 20 years and 4 months, and is bought on a 6% basis. Find the purchase price.

4. A \$1,000, 5% bond, with interest payable semiannually, is redeemable at par in 8 years and 5 months, and is bought on a $4\frac{1}{2}$% basis. Find the cash price.

5. Five \$1,000, 6% bonds, with interest payable semiannually, are due in 4 years and 4 months, and are purchased on a $4\frac{1}{2}$% basis. Find the value of the bonds.

Bonds to be redeemed above par. A form of bond which is redeemable (usually at the option of the maker) at a premium is a callable bond.

To find the value of a bond redeemable above par at the option of the maker, before it is due, it is necessary to determine the value:

(1) On the assumption that the bond will be redeemed at the optional redemption date, and at the optional price.

(2) On the assumption that the maker will not redeem the bond until maturity, and that it will then be redeemed at the par value.

After the two prices have been found, the purchaser should pay the lower of the two prices.

Example

A \$1,000, 6% bond, with interest payable semiannually, is due in 15 years, but the maker has the option of redeeming it at the end of 10 years at 105. What price should an investor pay for the bond, if he purchases it on a 5% basis?

(1) Calculation on the assumption that the bond will be redeemed at the end of 10 years at 105:

Formula

$$C'v^n + Cr \cdot a_{\overline{n}|i} = P$$

Arithmetical Substitution

$$1{,}050\left(\frac{1}{(1.025)^{20}}\right) + 30\left(\frac{1 - \dfrac{1}{(1.025)^{20}}}{.025}\right) = \$1{,}108.45.$$

Solution, Part 1

$(1.025)^{20}$ = 1.638616, compound amount of 1 at $2\frac{1}{2}$% for 20 periods

$1 \div 1.638616$ = .61027, present value of 1 at $2\frac{1}{2}$% for 20 periods

$\$1{,}050 \times .61027$ = \$640.78, present value of redemption price of bond

Solution, Part 2

$.61027$ = present value of 1 at $2\frac{1}{2}\%$ for 20 periods
$1 - .61027 = .38973$, compound discount on 1 at $2\frac{1}{2}\%$ for 20 periods
$.38973 \div .025 = 15.5892$, present value of annuity of 1
$\$30 \times 15.5892 = \$467.67$, present value of annuity of \$30

Solution, Part 3

$\$640.78 + \$467.67 = \$1{,}108.45$, value based on optional redemption

(2) Calculation on the assumption that the bond will not be paid until maturity:

Formula

$$Cv^n + Cr \cdot a_{\overline{n}|i} = P$$

Arithmetical Substitution

$$1{,}000\left(\frac{1}{(1.025)^{30}}\right) + 30\left(\frac{1 - \dfrac{1}{(1.025)^{30}}}{.025}\right) = \$1{,}104.65.$$

Solution, Part 1

$(1.025)^{30} = 2.097567$, compound amount of 1 at $2\frac{1}{2}\%$ for 30 periods
$1 \div 2.097567 = .476742$, present value of 1 at $2\frac{1}{2}\%$ for 30 periods
$\$1{,}000 \times .476742 = \$476.74$, present value of par of bond

Solution, Part 2

$.476742$ = present value of 1 at $2\frac{1}{2}\%$ for 30 periods
$1 - .476742 = .523258$, compound discount on 1 at $2\frac{1}{2}\%$ for 30 periods
$.523258 \div .025 = 20.93029$, present value of an annuity of 1
$\$30 \times 20.93029 = \$627.91$, present value of an annuity of \$30

Solution, Part 3

$\$476.74 + \$627.91 = \$1{,}104.65$, price of bond based on par

A comparison of the two results shows:

Value based on 15-year redemption price	\$1,108 45
Value based on 20-year redemption price	1,104.65

Therefore, the purchaser should pay the lower price, or \$1,104.65.

Problems

1. A \$1,000, $5\frac{1}{2}\%$ bond, with interest payable semiannually, matures in 10 years, but the company has the option of redemption at the end of 5 years at 104. What price should an investor pay for the bond, if he purchases it on a 5% basis?

2. A \$1,000, 5% bond, with interest payable semiannually, matures in 20 years, but the company has the option of redeeming the bond at the end of

15 years by paying a bonus or premium of 10%. What price should an investor pay for five of these bonds, if he purchases them on a 4% basis?

3. A $1,000, $5\frac{1}{2}$% bond, with interest payable annually, is redeemable in 5 years with a bonus of 10%. What price should be paid for this bond by a purchaser who desires to realize 6% on his investment? (v^5 at 6% = .7473.) Construct a table of verification.

Prepare the formula, solution, and verification for each of the following:

	Effective Interest	*Coupon Interest*	*Amount of Bond*	*Redeemable Value*	*Time to Run*	*Coupons Payable*
4.	5%	6%	$ 100	104	6 years	Annually
5.	6%	8%	1,000	105	$4\frac{1}{3}$ "	Annually
6.	5%	6%	500	108	$4\frac{1}{6}$ "	Semiannually
7.	6%	4%	500	110	$5\frac{1}{4}$ "	Semiannually

Serial redemption bonds. Many public and private corporations desire to pay off a portion of their bonds each year instead of setting up a sinking fund. Serial redemption bonds may be redeemed in equal or unequal periodic amounts.

If the bonds are not redeemed in equal periodic amounts, it is difficult to derive a formula or plan by which computations may be shortened or systematized. But if the redemptions are to be made in equal amounts and at regular periodic dates, formulas and solutions for finding the value of such an issue may be derived.

For the purpose of finding the present value of bonds to be redeemed in a series, it is well to analyze the issue into its component parts, and to calculate the value of each component part separately. The following example will illustrate this point:

Example

What is the present value of a bond issue of $10,000 bearing 5% interest, payable annually? These bonds are to be redeemed serially, $2,000 each year. Money is worth 6%, effective interest.

ANALYSIS OF THE CALCULATION OF THE VALUE OF A SERIAL REDEMPTION BOND

	First Period	*Second Period*	*Third Period*	*Fourth Period*	*Fifth Period*	*Multiplied by Present Value of Annuity of 1*	*Present Value of Annuity of Principal*	*Present Value of a Series of Annuities of Interests*
Principal	$2,000	$2,000	$2,000	$2,000	$2,000	$4 212363	$8,424.73	
Interest payments	100					94339		$ 94 34
" "	100	100				1 83339		183.34
" "	100	100	100			2 67301		267.30
" "	100	100	100	100		3.46511		346.51
" "	100	100	100	100	100	4.21236		421 24
							$8,424.73	$1,312 73

Summary

Present value of annuity of $2,000 at 6%.............	$8,424.73
Present value of 5 annuities of $100 each (the number of rents varies from 1 to 5, as shown above) at 6%	1,312.73
Present value of the serial redemption bonds..... ...	$9,737.46

A study of the above analysis shows two general divisions:

(1) The calculation of the present value of an annuity, the rents of which are the same as the amounts of the bonds which are redeemed annually.

(2) The calculation of the present value of a series of annuities, the rents of which are the same as the periodic interest payments on each series of bonds.

The first part needs no explanation, since the computation is similar to that of the present value of an ordinary annuity.

The sum of the values of the second part may be found by calculating the value of each separate annuity and then adding these values.

Let C = the par value of the bond;
R = annual amount redeemed;
r = the rate named in the bond, that is, the coupon or cash rate;
i = the rate of yield expected;
n = the number of years to maturity;
N = the number of equal redemptions to retire the issue;
P = the cost of the bond.

Procedure, Part 1: (*a*) Find the present value of an annuity of 1 at the effective rate per cent for as many periods as the bond has serial payments, $a_{\overline{n}|i}$.

(*b*) Multiply the present value of the annuity of 1 found in (*a*) by the number of dollars to be paid each period on the principal of the bonds, $Ra_{\overline{n}|i}$.

Procedure, Part 2: (*a*) Use the present value of an annuity of 1 found in Part 1 (*a*).

(*b*) From the number of annuities subtract the present value of an annuity of 1 found in (*a*), $N - a_{\overline{n}|i}$.

(*c*) Divide the annuity discount found in (*b*) by the effective rate to obtain the cumulative present worth of the annuities, $\dfrac{N - a_{\overline{n}|i}}{i}$.

(*d*) Multiply the average annuity price of 1 by the number of dollars in each interest payment or rent. The result will be the value of a series of annuities, $Cr\left(\dfrac{N - a_{\overline{n}|i}}{i}\right)$.

Procedure, Part 3: Add the result found in 1 (*b*) to that found in 2 (*d*).

Formula

$$Ra_{\overline{n}|i} + Cr\left(\frac{N - a_{\overline{n}|i}}{i}\right) = P$$

Arithmetical Substitution

$$2{,}000\left(\frac{1-\frac{1}{(1.06)^5}}{.06}\right)+100\left(\frac{5-\frac{1-\frac{1}{(1.06)^5}}{.06}}{.06}\right)=\$9{,}737.46.$$

Solution, Part 1

$(1.06)^5 = 1.3382256$, compound amount of 1 at 6% for 5 periods

$1 \div 1.3382256 = .7472582$, present value of 1 at 6% for 5 periods

$1 - .7472582 = .2527418$, compound discount on 1 at 6% for 5 periods

$.2527418 \div .06 = 4.2123638$, present value of annuity of 1 at 6% for 5 periods (See Table 5, page 532.)

$\$2{,}000 \times 4.2123638 = \$8{,}424.73$, present value of annuity of \$2,000

Solution, Part 2

$4.2123638 =$ present value of annuity of 1 at 6% for 5 periods

$5 - 4.2123638 = .7876362$, annuity discount on 5 annuities

$.7876362 \div .06 = 13.12727$, cumulative present value of annuities

$\$100 \times 13.1273 = \$1{,}312.73$, cumulative present value of annuity of \$100

Solution, Part 3

$\$8{,}424.73 + \$1{,}312.73 = \$9{,}737.46$, sum of present value of annuities

Problems

1. Compute the purchase price of \$5,000 of serial bonds, issued January 1, 1934, with 5% interest, payable annually, and dated to mature in five equal annual installments. Money is worth $5\frac{1}{2}$%.

2. Verify the solution of the above problem by setting up a columnar table showing: (*a*) date of maturity; (*b*) bonds outstanding; (*c*) coupon interest each year; (*d*) effective interest each year; (*e*) accumulation of discount; (*f*) carrying value.

3. Compute the purchase price of \$50,000 of serial bonds, issued January 1, 1934, bearing 5% interest, coupons due annually. These bonds were to mature serially in equal annual payments, beginning January 1, 1935, and each year thereafter for 10 years. Money was worth 4%.

4. Prepare a columnar table for Problem 3.

5. A \$20,000 serial bond issue, with interest at $5\frac{1}{2}$%, payable semiannually, is redeemable in ten equal semiannual installments. Money is worth 5%, convertible semiannually. Set up a columnar table similar to that in Problem 2.

Frequency of redemption periods. In the example on page 385, bonds were redeemed at each interest date. It is more usual, however, to find that the interest is payable semiannually, while the bond redemptions occur once a year.

ANALYSIS OF THE CALCULATION OF THE VALUE OF A SERIAL REDEMPTION BOND ISSUE, WITH INTEREST PAYMENTS, WHEN INTEREST IS PAYABLE EACH SIX MONTHS

	Assumed Dates						Present Value of an Annuity of 1 at 6.09%	Present Value of an Annuity of 1 at 3%	
	July 1	*Jan. 1*	*July 1*	*Jan. 1*	*July 1*	*Jan. 1*			
Principal of bond		$2,000		$2,000		$2,000	$2 66857		$5,337.14
" " "		25		25		25	2.66857		66 71
Semiannual interest	$25							$ 97087	24 27
" "	25	25						1.91346	47 84
" "	25	25	$25					2 82861	70 71
" "	25	25	25	25				3 71709	92.93
" "	25	25	25	25	$25			4 57970	114 49
" "	25	25	25	25	25	25		5.41719	135.43
									$5,889.52

Summary

Present value of annuity of $2,000 at 6.09%	$5,337 14
Present value of annuity of $25 at 6.09%	66.71
Present value of six rents of $25 each at 3%, interest compounded semiannually	485 67
	$5,889.52

Example

A $6,000 issue of serial bonds is to be redeemed in equal installments of $2,000, on January 1 of each year. The coupons are at the rate of 5%, payable semiannually. If money is worth 6%, interest convertible semiannually, what is the present value of the issue? (For the purpose of illustration, short-term bonds are used.)

A study of the analysis on page 388 shows three divisions:

(1) The calculation to find the present value of an annuity of $2,000, the annual payment on the bond issue, at the effective annual rate of 6.09%.

(2) The calculation to find the present value of an annuity of $25, the amount of the interest payment due at the end of each year.

(3) The calculation to find the present value of the series of six rents of $25 each at 3% semiannual interest.

It should be noticed that one of the interest payments of $25 due at the end of each year is separated from the other interest payments. This separation is made to reduce the remaining interest payments to a series of annuities of regularly increasing terms.

The computation may be shortened if the annual bond redemption payment of $2,000 and the annual interest payment of $25 are combined. It is then necessary to find the present value of an annuity the rents of which are $2,025 at 6.09% per annum. The formula and solution are derived by a method similar to that explained on page 386.

Let g represent the value of one coupon on the periodic cash interest on one bond.

Formula

$$(R + g) \cdot a_{\overline{n}|i} + g\left(\frac{N - a_{\overline{n}|i}}{i}\right) = P$$

in which $a_{\overline{n}|i}$ in the first part of the formula is calculated at the effective annual rate.

Arithmetical Substitution

$$2{,}025\left(\frac{1 - \dfrac{1}{(1.0609)^3}}{.0609}\right) + 25\left(\frac{6 - \dfrac{1 - \dfrac{1}{(1.03)^6}}{.03}}{.03}\right) = \$5{,}889.52.$$

Solution, Part 1

$(1.03)^2 = 1.0609$, effective ratio of increase
$1.0609 - 1 = .0609$, effective rate per annum
$(1.0609)^3 = 1.19405228$, compound amount of 1 at 6.09% for 3 periods

$1 \div 1.19405228 = .8374843$, present value of 1 at 6.09% for 3 periods

$1 - .8374843 = .1625157$, compound discount on 1 at 6.09% for 3 periods

$.1625157 \div .0609 = 2.668568$, present value of an annuity of 1 at 6.09% for 3 periods

$\$2{,}025 \times 2.668568 = \$5{,}403.85$, present value of annuity of \$2,025 at 6.09% for 3 periods

Solution, Part 2

$(1.03)^6 = 1.19405228$, compound amount of 1 at 3% for 6 periods

$1 \div 1.19405228 = .8374843$, present value of 1 at 3% for 6 periods

$1 - .8374843 = .1625157$, compound discount on 1 at 3% for 6 periods

$.1625157 \div .03 = 5.4171914$, present value of an annuity of 1 at 3% for 6 periods

$6 - 5.4171914 = .5828086$, annuity discount on 6 rents

$.5828086 \div .03 = 19.42695$, present value of 6 rents of 1 at 3% for 6 periods

$\$25 \times 19.42695 = \$485.67$, present value of 6 rents of \$25 each

Solution, Part 3

$\$5{,}403.85 + \$485.67 = \$5{,}889.52$, value of serial bond

Alternative solution. The following method of solving the preceding example is preferred when the number of redemptions are few. The procedure is that of finding the present value of the respective payments of interest and principal.

July 1	Interest	\$ 150 × .970873 =	\$ 145 63
Jan. 1	Int. and Principal	2,150 × .942595 =	2,026 58
July 1	Interest	100 × .915141 =	91 51
Jan. 1	Int. and Principal	2,100 × .888487 =	1,865 82
July 1	Interest	50 × .862608 =	43 13
Jan. 1	Int. and Principal........	2,050 × .837484 =	1,716 84
Total		=	\$5,889 51

Bonds redeemed by other than equal annual payments. If the bonds are to be redeemed in any other way than by equal annual payments beginning at the end of the first year, an analysis must be made of the component parts, and computations made for each part separately.

Example

What is the value of an issue of 5% serial redemption bonds for \$10,000, if \$2,000 is to be redeemed at the end of 6 years and \$2,000 at the end of each year thereafter until all the bonds have been redeemed? Interest coupons are payable semiannually. Money is worth 6%, interest converted semiannually.

ANALYSIS OF A SERIES OF SERIAL REDEMPTION BONDS, WITH INTEREST PAYMENTS

	Semiannual Periods																			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
												2,000		2,000		2,000		2,000		2,000
1st	25	25	25	25	25	25	25	25	25	25	25	25								
Series	25	25	25	25	25	25	25	25	25	25	25	25								
2nd	25	25	25	25	25	25	25	25	25	25	25	25	25	25						
Series	25	25	25	25	25	25	25	25	25	25	25	25	25	25						
3rd	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25				
Series	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25				
4th	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25		
Series	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25		
5th	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25
Series	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25
Totals	250	250	250	250	250	250	250	250	250	250										

From the above analysis there are found to be four component parts in the example:

(1) An annuity of \$2,000, the payments of which are to begin at the end of the sixth year and are to be made at the end of each year thereafter, at an effective interest rate of 6%, convertible semiannually, or 6.09% annually. This series of annuities is deferred for 5 years.

(2) An annuity, the rents of which are \$25, payable at the end of the sixth year, and annually thereafter, at 6.09%.

(3) A series of ten annuities of unequal length, the rents of which are \$25, payable at the end of each half-year. Each annuity is for 1 period longer than the preceding one. The rate of effective interest is 3%. This series of annuities is deferred for 5 years.

(4) A series of ten annuities of equal length, the rents of which are \$25, payable at the end of each period, with interest at 3%.

Formulas will not be given for this example, which does not involve anything new, but only the combining of certain principles already explained.

To shorten the calculation, (1) and (2) above can be combined, making a deferred annuity the rents of which are \$2,025 for 5 periods at 6.09%.

Procedure: (*a*) Find the present value at the beginning of the sixth year of (1) and (2) combined.

(*b*) Find the present value of (3) at the beginning of the sixth year.

(*c*) Multiply the sum of the present values found in (*a*) and (*b*) by the present worth of 1 for ten periods at 3%. The result is the

present value at the beginning of the first period of all the bonds and interest payments falling due after the end of the fifth year.

(*d*) Calculate the present value of (4); this result is the value at the beginning of the first year.

(*e*) Add the results found in (*c*) and (*d*). Their sum is the present value of the series of serial redemption bonds, with all the interest payments.

Solution, Part 1

NOTE: The compound amount and present value of $(1.0609)^5$ are the same as the compound amount and present value of $(1.03)^{10}$.

$(1.0609)^5 = 1.3439164$, compound amount of 1 for 5 periods at 6.09%

$1 \div 1.3439164 = 7440939$, present value of 1 for 5 periods at 6.09%

$1 - .7440939 = .2559061$, compound discount on 1 for 5 periods at 6.09%

$.2559061 \div .0609 = 4.2020707$, present value of an annuity of 1 for 5 periods at 6.09%

$\$2,025 \times 4.2020707 = \$8,509.19$, value of an annuity of \$2,025 for 5 periods at 6.09%

Solution, Part 2

$.7440939 =$ present value of 1 for 10 periods at 3% (same as 1 for 5 periods at 6.09%)

$1 - .7440939 = .2559061$, compound discount on 1 for 10 periods at 3%

$.2559061 \div .03 = 8.530202$, present value of an annuity of 1 for 10 periods at 3%

$10 - 8.530202 = 1.469798$, annuity discount on the series of 10 annuities of 1

$1.469798 \div .03 = 48.9932$, present value of the series of 10 annuities of 1

$\$25 \times 48.9932 = \$1,224.83$, present value of the series of 10 annuities of \$25 each

Solution, Part 3

$\$8,509.19 + \$1,224.83 = \$9,734.02$, amount of present value at the beginning of the sixth year

$.7440939 =$ present value of 1 for 10 periods at 3%

$\$9,734.02 \times .7440939 = \$7,243.02$, present value of serial bonds and of the series of 10 annuities

Solution, Part 4

$8.530202 =$ present value of an annuity of 1 for 10 periods at 3%

$\$25 \times 10 = \$250$, sum of the 10 payments at the end of each period

$\$250 \times 8.530202 = \$2,132.55$, present value of the annuity of \$250 for 10 periods at 3%

Solution, Part 5

$7,243.02 + $2,132.55 = $9,375.57, present value of serial bonds, with all interest payments

VERIFICATION OF CALCULATION OF THE VALUE OF A SERIES OF SERIAL REDEMPTION BONDS

End of Period	*Bonds Redeemed*	*Effective Interest*	*Coupon Interest*	*Amortization of Discount*	*Bonds Less Discount*
					$9,375.57
1		$281.27	$250.00	$31.27	9,406.84
2		282.20	250.00	32.20	9,439.04
3		283.17	250.00	33.17	9,472.21
4		284.17	250.00	34.17	9,506.38
5		285.19	250.00	35.19	9,541.57
6		286.25	250.00	36.25	9,577.82
7		287.33	250.00	37.33	9,615.15
8		288.45	250.00	38.45	9,653.60
9		289.61	250.00	39.61	9,693.21
10		290.80	250.00	40.80	9,734.01
11		292.02	250.00	42.02	9,776.03
12	$2,000.00	293.28	250.00	43.28	7,819.31
13		234.58	200.00	34.58	7,854.89
14	2,000.00	235.62	200.00	35.62	5,889.51
15		176.69	150.00	26.69	5,916.20
16	2,000.00	177.49	150.00	27.49	3,943.69
17		118.31	100.00	18.31	3,962.00
18	2,000.00	118.86	100.00	18.86	1,980.86
19		59.43	50.00	9.43	1,990.29
20	2,000.00	59.71	50.00	9.71	0.00

Problems

1. The State Highway Department of Michigan desires to know the value of a series of road bonds which it is about to issue. The bonds will have a par value of $200,000, and will bear 5% interest, payable semiannually. They are to be redeemed serially, in installments of $40,000. The first redemption payment is to be made one year from the date of issue, and the other payments are to be made annually thereafter. Money is worth 6%, interest compounded semiannually. Find the value of the bonds, and draw up a table of analysis as a proof of your solution.

2. A corporation wishes to float an issue of serial bonds for $100,000. These bonds are to be redeemed in yearly installments of $20,000, the first redemption payment to be made at the beginning of the sixth year. The interest rate is 5%, payable semiannually, and the effective rate is $4\frac{1}{2}$%, interest convertible semiannually. Draw up a table of analysis. From the analysis, prepare the formula and solution.

3. An issue of serial bonds bearing 4% interest, payable semiannually, is to be redeemed serially in installments of $4,000. The first redemption payment is to be made at the end of the fifth year, and the other payments are to be made annually thereafter. At what price must $20,000 of these serial bonds be purchased in order to net the purchaser 5% annual effective interest?

4. An issue of \$50,000 of bonds bearing interest at 5%, payable semiannually, is sold to produce $5\frac{1}{2}$% effective interest, convertible semiannually. The bonds are to be retired serially, as follows:

\$10,000 at 104 in 6 years
10,000 at 103 in 7 years
10,000 at 102 in 8 years
10,000 at 101 in 9 years
10,000 at par in 10 years

Set up a schedule in columnar form, showing the book value, cash interest, effective interest, amortization of discount, and par value of bonds outstanding each year.

Bonds redeemable from a fund. Frequently, bonds are redeemed periodically from a fund; that is, as soon as money is put into the fund, or when money becomes available at the end of an interest period, it is at once used to redeem outstanding bonds. No difficulty would be encountered in making the computations necessary in this system of redemption, except for the fact that bonds are usually issued in denominations of \$100, \$500, or \$1,000, and the payments into the fund may exceed the expenditures from the fund for the redemption of bonds. A residue would then be left in the fund, and this residue should earn interest.

Example

The X.Y.Z. Company issues \$100,000 of bonds, par value \$100, and sets up a sinking fund for their periodic redemption. The bond interest and the sinking fund interest are each 6%. The bonds are to run for 5 years, with interest payable semiannually, and are to be kept alive in the treasury. Sinking fund payments are to be made semiannually. Interest is to be paid by the trustee on the balance remaining in the fund. Show: (*a*) the periodic payments into the sinking fund; (*b*) the interest accrued periodically on bonds redeemed; (*c*) the total amount that must be invested in the sinking fund each period; (*d*) the amount of bonds purchased periodically for the sinking fund; (*e*) the cash balance held in the treasury; and (*f*) the interest on the cash balance held in the treasury.

Let: i = the effective interest per period.
And: n = the number of periods.

Formula	*Arithmetical Substitution*
$\frac{1}{a_{\overline{n}\rceil i}} \cdot P = R$	$\frac{1}{\frac{1 - \frac{1}{(1.03)^{10}}}{.03}} \times \$100{,}000 = \$11{,}723.05$

From Table 6, page 534, the value of $\frac{1}{\frac{1 - \frac{1}{(1.03)^{10}}}{.03}}$ is found to be .1172305,

and .1172305 × \$100,000 = \$11.723.05.

The remaining part of the solution can be derived from the following table:

Periodic Payment	*Balance from Preceding Period*	*Total Amount Available*	*Bond Interest*	*Bonds Redeemed*	*Cash Balance*	*Interest on Cash Balance*	*Bonds Outstanding*
							$100,000
$11,723 05		$11,723.05	$3,000	$ 8,700	$23 05	$ .69	91,300
11,723 05	$23 74	11,746 79	2,739	9,000	7 79	23	82,300
11,723 05	8 02	11,731 07	2,469	9,200	62 07	1 86	73,100
11,723 05	63.93	11,786 98	2,193	9,500	93 98	2 82	63,600
11,723 05	96 80	11,819 85	1,908	9,900	11 85	.36	53,700
11,723 05	12 21	11,735 26	1,611	10,100	24 26	.73	43,600
11,723 05	24 99	11,748.04	1,308	10,400	40 04	1.20	33,200
11,723 05	41 24	11,764 29	996	10,700	68 29	2.05	22,500
11,723 05	70 34	11,793 39	675	11,100	18.39	.55	11,400
11,723 06	18.94	11,742.00	342	11,400			

In the above example, the first payment into the fund is $11,723.05. The payment on the interest will be $3,000, leaving $8,723.05 to be used for the redemption of bonds. As the bonds issued are of $100 denomination, only $8,700 of this fund can be used for the redemption of bonds, leaving a balance of $23.05 cash in the hands of the trustee.

In order to verify the solution of a problem of this kind, it is necessary to charge the trustee with interest at the sinking fund rate on the balance remaining in his hands. The interest on $23.05, the cash balance in the trustee's hands, for 6 months at 3% is $.69. This interest added to the cash balance gives the balance from the preceding period.

Problems

1. A $200,000 bond issue, maturing in 8 years, bears interest at 6%, payable semiannually. The par value of the bonds is $1,000. A sinking fund is set up, and the trustee is to purchase bonds semiannually at par and keep them alive in the treasury. Money is worth 6%, interest convertible semiannually. Prepare a table, showing: (*a*) the semiannual periodic payments to the sinking fund; (*b*) the interest accrued on the bonds; (*c*) the par value of the bonds purchased semiannually; (*d*) the cash balance in the hands of the trustee; and (*e*) the interest on the cash balance.

2. An issue of $200,000 of 6%, 10-year bonds is floated by a corporation. The par value is $100, and the interest coupons are payable semiannually. The bonds are to be redeemed semiannually, and the sinking fund payments necessary for the redemption of the principal and the payment of the interest are placed in the hands of the trustee. The sinking fund rate is 6%. Prepare a table, as in Problem 1.

Effective rate of interest on bonds. One question which is very important to the investor is, "What rate of interest will I receive on this bond?" or "What will be the effective rate on the

money invested?" The investor knows the amount of each interest coupon, but the coupon rate is based on par value and not on the amount of money which has been invested.

Because of the fact that the unknown effective interest rate must be used more than once in the algebraic formula for the calculation of the price of a bond, the absolute effective rate is difficult to find.

However, two methods which give a close approximation may be used. The rate may be found:

(1) By the use of bond tables, or test rates, and interpolation.

(2) By formulas specially constructed to give approximately the required rate. Most formulas, and sometimes a method of averages, will give the rate accurately enough for ordinary purposes.

Effective rate on bonds sold at a premium.

Example

If a $100, 3-year, 6% bond, bearing semiannual interest coupons, is bought at $105.38, what rate of interest will be realized on the investment?

SECTION OF BOND TABLE

CASH INTEREST PAYABLE SEMIANNUALLY

Effective Interest Rate	*3%*	*3½%*	*4%*	*4½%*	*5%*	*6%*	*7%*
3 75	$97 89	$99 30	$100 70	$102.11	$103.52	$106 33	$109 14
3 80	97.75	99.16	100 56	101 97	103.37	106 18	108 99
3.875	97.54	98 95	100 35	101 75	103 16	105 96	108.77
3 90	97 48	98 88	100.28	101.68	103 09	105 89	108 70
4.00	97.20	98 60	100 00	101.40	102 80	105.60	108 40
4 10	96 92	98 32	99 72	101 12	102 52	105 31	108 11
4.12½	96.86	98.25	99 65	101 05	102.45	105 24	108 04
4.20	96.65	98.05	99 44	100 84	102.23	105.02	107 82
4.25	96.51	97.91	99.30	100.70	102.09	104.88	107 67

Solution

In the 6% column of the section of the bond table given above will be found, opposite 4%, the price of $105.60, and opposite 4.1%, the price of $105.31. The rate is therefore somewhere between 4% and 4.1%. A more exact approximation may be determined as follows:

Interpolation

Value on a basis of 4%	$105 60
Value on a basis of 4.1%	105 31
Difference in value caused by a difference in rate of .1%	$ .29
Value on a basis of 4%	$105 60
Price paid	105.38
Difference between price paid and value on a basis of 4%	$ .22

If the difference between the price paid and the value on a basis of 4% is $.22, and the difference in the value caused by a difference of .1% in the rate is $.29, the rate earned will be $\frac{22}{29}$ of .1% greater than 4%, or 4.076%.

Effective rate on bonds sold at a discount.

Example

What will be the rate of income on an investment in a 4%, semiannual, 3-year bond bought at $99.35?

Solution

By reference to the table above, the computation may be made as follows:

Value of a 4% bond at 4.2% effective interest . . .	$99 44
Value of a 4% bond at 4.25% effective interest	99 30
Difference in value caused by a difference in rate of .05%	$ 14
Value on a basis of 4.2%..........................	$99 44
Price of bond......................................	99 35
Difference	$ .09
$\frac{9}{14}$ of .05%...	.032%
4.20% + .032%..................................	4.232%

Hence, the approximate rate is 4.232%.

Computation when bond table is not available. If no bond table is available, an approximate rate may be computed by the use of test rates, but care must be exercised in the choice of the rates, which must be as close to the actual rate as it is possible to estimate.

Example

If a $100, 5%, 5-year bond, with interest payable semiannually, is bought at $97.31, what rate of interest will be realized on the investment?

Solution

The first step is to find the cost at an estimated effective rate. Assume this rate to be $5\frac{1}{2}$%. By the formula, previously given, for finding the purchase price of bonds, this bond at 5.5% is worth $97.84. As this price is higher than the price paid, the rate is too low, and it is necessary to try a higher rate. Make a second trial at 5.75%. The purchase price is then found to be $96.78. As one price is slightly above and the other slightly below the actual price paid, the approximate rate can be found by interpolation.

Interpolation

Value on a basis of 5.50%............................	$97 84
Value on a basis of 5.75%	96 78
Difference in value caused by a difference in rate of $\frac{1}{4}$%..	$ 1.06
Value on a basis of 5.50%	$97 84
Price paid for bond	97 31
Difference between price at 5.50% and price paid...... .	$ .53

Since the difference between the price paid and the price at 5.50% is $.53, and the difference between the value on a 5.50% basis and the value on a 5.75% basis is $1.06, the rate earned will be approximately 5.50% plus $\frac{53}{106}$ of the difference between 5.50% and 5.75%, or 5.625%.

Approximation by averages. A fair approximation may be made by the use of averages. In the above example, the rate would be found as shown on the next page.

Semiannual yield	$ 2.50
Total gain on redemption, $100 − $97.31 = $2 69	
Average gain on redemption ($2.69 ÷ 10)........	.269
Average yield per period	$ 2 769
Capital invested at beginning of 10th period before maturity	$ 97 31
Capital invested at beginning of last period before maturity ($100 − .269)	99 731
Total	$197 041
Dividing by 2, to find average capital	$ 98 5205
Average yield ÷ average capital .	Rate
2.769 ÷ 98.5205	02811
Multiplying by 2 ..	05622, or 5 622%
Correct yield	0562468, or 5.625%

Problems

In each of the following, find the effective rate by two methods: (1) by the use of bond tables, or test rates, and interpolation; (2) by the use of averages.

	Purchase Price	*Nominal Rate*	*Par Value*	*Time in Years*	*Interest Payable*	*Effective Rate Method 1*	*Effective Rate Method 2*
1.	$ 92 00	4%	$ 100 00	3	Semiannually		
2.	1,015 00	4%	1,000 00	9	"		
3.	$9,750 00	4%	10,000 00	29	"		
4.	545 00	5%	500 00	8	"		
5.	925 00	5%	1,000 00	15	"		
6.	983 75	5%	1,000 00	18	"		
7.	9,732 50	5½%	10,000 00	5	"		
8.	5,201 50	5½%	5,000 00	9	"		
9.	96 85	6%	100 00	4	"		
10.	73 55	6%	100 00	6	"		

Amortization of discount, premium, or discount and expense on serial redemption bonds. The calculation of the amortization of bond discount, bond premium, or bond discount and expense, when bonds are to be redeemed serially, or in unequal amounts, is a problem which requires particular attention, since the distribution over the period of years must be equitable. The two methods which are most commonly used are:

(1) The bonds outstanding method.

(2) The scientific or effective-interest method.

Bonds outstanding method. It would be incorrect to write off the discount or premium, or the discount and expense, on serial redemption bonds or on a bond issue which has no regular redemption period, by the straight-line method. In some cases it is difficult to calculate the portion to be amortized by the scientific method. Because of the ease of the calculations and the fair

degree of accuracy which it affords, the bonds outstanding method is the one most commonly used.

Procedure: (*a*) Find the sum of the bonds outstanding during each period of the life of the bond issue.

(*b*) Use the sum found in (*a*) as the denominator of a series of fractions, and use the sum of the bonds outstanding each period as successive numerators. The sum of all these fractions, of course, will in every problem be equal to 1.

(*c*) Multiply the total bond discount or the premium, or the total bond discount and expense, by the appropriate fraction, to obtain the portion of discount or premium to be amortized each period.

Example

An issue of ten bonds of $1,000 each, bearing 5% interest, payable semi-annually, is to be redeemed as follows: $3,000 at the end of the sixth year; $3,000 at the end of the eighth year; and $2,000 at the end of each year thereafter. The bonds are sold at a discount of $400. Compute an equitable amortization of the discount over the life of the bonds.

Solution

Periods of One-half Year Each	*Bonds Outstanding*	*Fraction*	*Amortization Written Off*	*Discount on Bonds*	*Bonds Less Discount*
	$10,000 00			$400 00	$9,600 00
1	10,000 00	10/160	$ 25 00	375 00	9,625 00
2	10,000.00	10/160	25 00	350 00	9,650.00
3	10,000 00	10/160	25 00	325 00	9,675 00
4	10,000 00	10/160	25 00	300 00	9,700 00
5	10,000 00	10/160	25 00	275 00	9,725 00
6	10,000 00	10/160	25 00	250 00	9,750.00
7	10,000 00	10/160	25 00	225.00	9,775 00
8	10,000.00	10/160	25 00	200.00	9,800.00
9	10,000.00	10/160	25 00	175.00	9,825 00
10	10,000 00	10/160	25 00	150 00	9,850 00
11	10,000 00	10/160	25 00	125 00	9,875.00
12	7,000 00	10/160	25 00	100 00	6,900 00
13	7,000 00	7/160	17 50	82 50	6,917 50
14	7,000 00	7/160	17 50	65 00	6,935 00
15	7,000 00	7/160	17 50	47 50	6,952 50
16	4,000 00	7/160	17 50	30 00	3,970 00
17	4,000 00	4/160	10 00	20 00	3,980 00
18	2,000.00	4/160	10 00	10 00	1,990 00
19	2,000 00	2/160	5.00	5 00	1,995 00
20	000 00	2/160	5.00	0.00	000 00
		160/160	400.00		

Scientific method. To find by the scientific method the amount of discount or premium to be amortized on an issue of serial redemption bonds, it is necessary to find the approximate effective rate of interest which these bonds will bear; and to find the approximate effective rate it is necessary to use other approximations.

Procedure: (*a*) Determine the average life of the bonds, in periods.

(*b*) Determine, by the use of a bond table or by annuity calculations, the approximate effective interest rate for one bond having a life of the same number of periods as the average life found in (*a*).

(*c*) From a bond table or by annuity calculations, find the value of one bond at each of the annual redemption dates, at the approximate effective rate found in (*b*).

(*d*) By using the values found in (*c*), find the total value of the bonds redeemed at each redemption date.

(*e*) Add the values found in (*d*).

(*f*) Compare the sum found in (*e*) with the actual price received for the bonds.

(*g*) By the same process, determine another approximate rate.

(*h*) By interpolation, determine the error, using the approximate rates found above.

The computation of the periodic amortization is comparatively simple when the cost of the serial redemption bonds, the nominal or coupon rate, and the effective rate of interest are known. Valuation of each member of the series is equally simple. Refer to the table on page 402, which shows the amortization of discount for a series of serial redemption bonds. The periodic amortization is the difference between the effective income and the actual cash income. It may be observed that the difficulty of the whole calculation is the determination of the effective rate.

The example that was previously given under the bonds outstanding method would be solved by the scientific method as follows:

Solution

(*a*)

Bonds	*Maturing In*	*Product*
3	6 years	18
3	8 "	24
2	9 "	18
2	10 "	20
10		80

80 ÷ 10 = 8, or an average life of 8 years

$9,600 ÷ 100 = $96, the average sales price per hundred

(*b*) Refer to a bond table which shows the values at different effective rates of a 5% bond maturing in 8 years, with interest payable semiannually. The price nearest to 96 is found to be 96.02. This amount is opposite the effective rate of $5\frac{5}{8}$%.

(*c*) The next step is to find the value at $5\frac{5}{8}$% of all the bonds in the series. To do this, refer to the bond table which shows a cash rate of 5% and an effective rate of $5\frac{5}{8}$%.

(*d*) The results will be:

Par Value of Bonds	*Years to Run*	*Value at $5\frac{5}{8}$%*	*Total Value*
$3,000 00	6	$96.85	$2,905 50
3,000 00	8	96 02	2,880.60
2,000 00	9	95 63	1,912 60
2,000.00	10	95.27	1,905 40
(*e*) Value of series			$9,604 10

(*f*) As the sales price of $9,600 for the bonds is below the price based on a $5\frac{5}{8}$% rate, the effective rate is a little larger than $5\frac{5}{8}$%. A test at $5\frac{3}{4}$% would result in the following:

(*g*)

Par Value of Bonds	*Years to Run*	*Value at $5\frac{3}{4}$%*	*Total Value*
$3,000 00	6	$96 24	$2,887 20
3,000 00	8	95 24	2,857 20
2,000 00	9	94 79	1,895 80
2,000 00	10	94 36	1,887 20
Value of series			$9,527 40

(*h*)

Interpolation

Value of series at $5\frac{5}{8}$%	$9,604 10
Value of series at $5\frac{3}{4}$%	9,527 40
Difference caused by a difference in rate of $\frac{1}{8}$%	$ 76.70
Value of series at $5\frac{5}{8}$%	$9,604.10
Sales price	9,600 00
Difference	$ 4.10
410/7670 of $\frac{1}{8}$%	.006682
Trial rate	5 625
Add fractional rate found	.006682
Effective rate	5.631682
Effective rate to be used semiannually (5.631682 ÷ 2)	2.815841%

TABLE SHOWING AMORTIZATION OF DISCOUNT FOR SERIES OF SERIAL REDEMPTION BONDS

End of Period	*Bonds Redeemed*	*Effective Rate, 2.815841%*	*Coupon Rate, 2.5%*	*Amortization of Discount*	*Bonds Less Discount*
					$9,600 00
1		$270.32	$250.00	$20 32	9,620 32
2		270 89	250 00	20 89	9,641 21
3		271 48	250 00	21 48	9,662.69
4		272 09	250 00	22 09	9,684 78
5		272.71	250 00	22 71	9,707 49
6		273 35	250 00	23 35	9,730 84
7		274 01	250 00	24 01	9,754 85
8		274 68	250 00	24.68	9,779 53
9		275 38	250.00	25 38	9,804 91
10		276 09	250 00	26.09	9,831 00
11		276 83	250 00	26 83	9,857 83
12	$3,000 00	277 58	250 00	27 58	6,885 41
13		193 88	175.00	18.88	6,904 29
14		194.41	175.00	19 41	6,923 70
15		194 96	175.00	19 96	6,943 66
16	3,000 00	195.52	175.00	20 52	3,964 18
17		111.63	100 00	11.63	3,975 81
18	2,000 00	111 95	100 00	11 95	1,987 76
19		55 97	50.00	5 97	1,993 73
20	2,000 00	56 14	50 00	6 14	.13*

* Error caused by approximation and by the use of bond tables having only 4 places.

Problems

1.* On January 1, 1934, a corporation floated a bond issue of $300,000 to be retired serially over a period of 8 years as follows:

Dec. 31, 1934 ...	$10,000	Dec. 31, 1938	$ 30,000
Dec. 31, 1935 . .	15,000	Dec. 31, 1939 .	35,000
Dec. 31, 1936 . . .	20,000	Dec. 31, 1940	40,000
Dec. 31, 1937	25,000	Dec. 31, 1941	125,000

The discount and expense of issuing the bonds amounted to $33,000.

Draft a schedule, showing how much of such bond discount and interest you would claim as a deduction from gross income for federal income tax purposes for each of the years 1934 to 1941, inclusive.

2.* A city wishes to buy new fire equipment. The cost will be $500,000, and the equipment will have an estimated life of 10 years, and no salvage value. It is necessary to issue bonds to pay for this purchase, although, at the present time, interest rates are high—6%, payable annually.

How would you suggest that these bonds be issued, and what will be the annual cost to the taxpayers?

It is expected that a sinking fund would not earn more than an average of 3%.

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* American Institute Examination.

The bonds will be issued in denominations of $100 and multiples thereof.
Given:

$(1.06)^9 = 1.679479$	$(1.06)^{10} = 1.790848$
$(1.03)^9 = 1.304773$	$(1.03)^{10} = 1.343916$

3.* A series of 5% bonds totalling $100,000, dated January 1, 1934, is redeemable at par by ten annual payments of $10,000 each, beginning December 31, 1944. What equal annual payments to a sinking fund are required to be provided on a 4% basis in order to pay off the bonds as they mature?

The first payment to the sinking-fund trustees is to be made on December 31, 1934, and the further payments are to be made annually thereafter.

What is the status of the sinking fund on December 31, 1943, 1944, and 1945?

Given at 4%:

$(1 + i)^{10} = 1.48024428$	$(1 + i)^{20} = 2.19112312$
$v^{10} = .67556417$	$v^{20} = .45638695$

Given at 5%:

$(1 + i)^{10} = 1.6288946$	$(1 + i)^{20} = 2.6532977$
$v^{10} = .6139133$	$v^{20} = .3768895$

4. On June 1, a corporation sold a $3,000,000 issue of 6%, first mortgage, 20-year bonds at a discount of $300,000. According to the terms of sale these bonds were to be retired at the rate of $150,000 each year, the purchases to be made in the open market. The first retirement in the amount of $75,000 was to be made on March 1 following the date of issue, and $75,000 was to be retired each six months thereafter.

Any premium paid or discount received on bonds purchased for retirement was to be added or deducted, whichever the case might be, to that year's portion of discount, and was to be amortized as shown by the schedule of amortization.

Prepare a schedule, showing the amortization of bond discount by the bonds outstanding method.

5. An issue of $175,000 of 6½% bonds was sold for 95, the amount of the discount being $8,750. Other expenses pertaining to the issue of the bonds amounted to $596.67, making the total of bond discount and expense $9,346.67. These bonds are dated March 30, 1944, and the due dates are as follows:

$ 3,500 due April 15, 1944	$ 12,000 due April 15, 1949
5,000 due April 15, 1945	14,500 due April 15, 1950
7,000 due April 15, 1946	15,000 due April 15, 1951
8,000 due April 15, 1947	100,000 due April 15, 1952
10,000 due April 15, 1948	

Prepare a schedule, showing the annual charge for amortization of bond discount and expense as of the close of each year, December 31.

6.† On April 1, 1934, Southern Railway Equipment Gold 4½'s, due serially on each successive coupon date in October and April between October 1, 1934, up to, and including, April 1, 1948, were sold to yield 4¾%. The Bank of

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* American Institute Examination.

† Moore, Justin H., *Handbook of Financial Mathematics.* New York, Prentice-Hall, Inc., 1929.

Montrose purchased 2 bonds due April 1, 1937, 4 due October 1, 1938, 1 due October 1, 1942, 7 due April 1, 1943, and 1 due April 1, 1945.

(*a*) What was the total price paid? (*b*) Set up a schedule showing the investor's situation in regard to the book value of this investment at the beginning of each six-month period until the final maturity.

7.* On November 1, 1933, an investor purchased 20 bonds, each with a par value of $1,000, and maturing as follows:

	Number of Bonds
May 1, 1934	1
Nov. 1, 1934	3
May 1, 1935	2
May 1, 1936	5
May 1, 1937	9

The price paid for the 20 bonds was $20,417.11. The coupon rate is 6%, payable May 1 and November 1.

(*a*) Determine the rate of yield. (*b*) Set up a schedule showing the investor's situation in regard to the book value of this investment at the beginning of each six-month period until the final maturity.

Review Problems

1. A serial issue of $20,000 in denominations of $1,000 with interest at 4% has maturity of $1,000 at each interest date, March 1 and September 1. Find the price at the date of issue, March 1, to yield the purchaser $3\frac{1}{2}\%$.

2. If, in Problem 1, the first maturity were to occur 2 years following the date of issue, and then each six months thereafter, find the price at the date of issue, March 1, that would yield the purchaser $3\frac{1}{2}\%$.

3. A $1,000 bond with interest at $4\frac{1}{2}\%$, maturing April 1, 1954 and redeemable at 102 on any interest date after January 1, 1945, was sold on July 21, 1942 on a 5% basis. What price was paid for it?

4. A loan of $5,000 at 5% payable semiannually is repayable on each interest date in installments of $1,250. Find the purchase price to yield the investor 4%.

5. A loan of $5,000 with interest at 6% payable semiannually is to run 5 years and then be redeemed in annual installments of $1,000. What is the purchase price to yield 5% convertible semiannually?

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* Moore, Justin H., *Handbook of Financial Mathematics* New York, Prentice-Hall, Inc., 1929.

CHAPTER 34

Asset Valuation Accounts

Asset valuation. At the moment that an asset is purchased for use, it is said to be worth cost. When this asset can no longer be used for the purpose for which it was purchased, and can be sold for little or nothing, it is said to be worth scrap value. The difference between the scrap value and the cost is the depreciation.

Depreciation. Depreciation is the decline in value of a physical asset caused by wear, tear, action of the elements, obsolescence, supersession, or inadequacy and resulting in an impairment of operating effectiveness.

Depletion. Depletion is the progressive extinction of a wasting asset by a reduction in the quantity. A coal mine is depleted year by year as the coal is mined.

Depreciation methods. There are many methods of calculating the periodic depreciation charge. Probably the following are the most serviceable:

(1) Straight-line method.
(2) Working-hours method, or unit-product method.
(3) Sum-of-digits method.
(4) Sinking-fund method.
(5) Annuity method.
(6) Fixed-percentage-of-diminishing-value method.

The accountant should know how to calculate depreciation by each method, should be able to discuss the advantages and disadvantages of each, and should be able to formulate tables of comparison showing the results of each.

Straight-line method. This is the simplest method, and is the one most commonly used.

Procedure: (*a*) Find the difference between the cost and the scrap value.

(*b*) Divide the difference found in (*a*) by the number of periods which the asset is expected to be of service. The result is the depreciation charge per period.

Example

What will be the depreciation charge and the asset valuation at the end of each year for an asset costing $1,000, and having an estimated life of 10 years and an estimated scrap value of $100?

Formula

Cost − Scrap = Depreciation
Depreciation ÷ Number of years = Annual charge

Arithmetical Substitution

$1,000 − $100 = $900
$900 ÷ 10 = $90

TABLE OF DEPRECIATION

Years	*Periodic Depreciation Charge*	*Accumulated Depreciation Reserve*	*Asset Value*
			$1,000
1	$90	$ 90	910
2	90	180	820
3	90	270	730
4	90	360	640
5	90	450	550
6	90	540	460
7	90	630	370
8	90	720	280
9	90	810	190
10	90	900	100

Problems

1. Set up a table showing the annual depreciation, carrying value, and accumulated depreciation for an asset costing $3,500, and having a life of 10 years and an estimated scrap value of $500.

2. Prepare tables showing by the straight-line method the depreciation on the following machines:

Assets	*Cost*	*Estimated Scrap Value*	*Estimated Life in Years*
Lathes	$5,000	$600	10
Milling machines	3,000	400	12
Power equipment.............	4,100	200	10
Furniture...................	600	150	15

3. The following fixed assets belong to the Western Hardware Company:

Asset	*Cost*	*Estimated Scrap Value*	*Estimated Life in Years*
Buildings ..	$100,000	$35,000	20
Machinery...............	70,000	25,000	15
Tools...................	20,000	5,000	10
Patterns.............	10,000	none	8

Compute the amount of annual depreciation by the straight-line method.

4. Complete the following schedule of fixed assets and depreciation, for the purpose of supporting an income tax return (allow six months' average or additions):

Asset	Previous Cost	Additions	Rate	Reserve	Current Depreciation
Land	$75,000 00	none	none	none	none
Brick buildings . .	92,519 61	$1,046 91	3%	$ 9,461 92	$ . . . . .
Wooden buildings	35,654 18	1,126 19	5%	12,319 14	. . . . .
Machinery and tools	51,252 19	9,217.62	10%	11,463 21	. . . . .
Office furniture	3,469 52	417 51	10%	1,121 44	. . .
Automobile trucks .	3,219 52	1,750 19	25%	1,749 32	. . . . . .
Spur track .	712 92	none	5%	128 34	.
	$	$.		$.	$ . . .

Working-hours or unit-product method. This method is based on the number of hours which the asset is in use, or on the number of units produced.

Example

A certain one-purpose machine which costs $1,000, and has no scrap value, has been installed in a factory. A machine of this class produces 10,000 units of product during its life. Assuming that the annual production is as given below, set up a table showing the depreciation to be written off each year.

First year	1,000 units	Fifth year	1,000 units
Second year	2,000 "	Sixth year	1,200 "
Third year	1,800 "	Seventh year	1,200 "
Fourth year	1,000 "	Eighth year . .	800 "

Solution

TABLE OF DEPRECIATION

Year	*Unit Fraction of Cost*	*Periodic Depreciation Charge*	*Accumulated Depreciation Reserve*	*Asset Value*
				$1,000
1	1,000/10,000	$100	$ 100	900
2	2,000/10,000	200	300	700
3	1,800/10,000	180	480	520
4	1,000/10,000	100	580	420
5	1,000/10,000	100	680	320
6	1,200/10,000	120	800	200
7	1,200/10,000	120	920	80
8	800/10,000	80	1,000	0

Problems

1. Show in appropriate form the yearly depreciation, accumulated depreciation, and asset value of a machine which cost $7,400, and which will have a scrap value of $200. Assume that machines of this class have a working-hour average life of 24,000 hours; also assume that the machine will be run as follows:

First year	2,000 hours	Sixth year	2,000 hours
Second year	2,000 "	Seventh year	3,000 "
Third year	1,800 "	Eighth year	3,000 "
Fourth year	2,600 "	Ninth year .	3,000 "
Fifth year	2,800 "	Tenth year	1,800 "

2. An aircraft motor costing $7,700, and having an estimated scrap value of $1,000, is assumed to have a useful life of 2,000 hours. If this motor is oper-

ated 350 hours during a certain month, what should be the charge for depreciation in that month?

Sum-of-digits method. Those who believe that the depreciation charge should be large during the early years of the useful life of the asset, will find the sum-of-digits method of value.

Procedure: (*a*) Find the sum of the digits, or numbers representing the periods of useful life of the asset. Use this sum as the denominator of certain fractions.

(*b*) Use the same digits or numbers in inverse order as the numerators of these fractions.

(*c*) Compute the periodic depreciation by multiplying the total depreciation by the fractions obtained in (*a*) and (*b*).

Example

An asset is valued at $1,000, and has a scrap value of $100. What should be the depreciation charges if the asset is to be written down in 9 years by the sum-of-digits method?

Solution

TABLE OF DEPRECIATION

Year	*Fractional Part*	*Periodic Depreciation Charge*	*Accumulated Depreciation Reserve*	*Asset Value*
				$1,000
1	9/45	$180	$180	820
2	8/45	160	340	660
3	7/45	140	480	520
4	6/45	120	600	400
5	5/45	100	700	300
6	4/45	80	780	220
7	3/45	60	840	160
8	2/45	40	880	120
9	1/45	20	900	100
45	45/45	$900		

The denominator of the fractions used in the second column is found by adding the first column. The numerators are the same numbers taken in inverse order.

Problem

Compute by the sum-of-digits method the depreciation charges, the asset valuation, and the depreciation provision for each year on each of the following machines:

Asset	*Cost*	*Scrap*	*Life of Asset, Years*
Power lathe	$1,300	$200	10
Hack saw	350	38	12
Turret lathe	3,000	200	7
Boiler	2,700	180	8
Power equipment	800	50	5
Delivery equipment	2,000	500	5

Present the information by means of tables

Sinking-fund method. This method is based on the assumption that a fund will be provided to replace the asset at the expiration of its life. Seldom is such a fund provided, but the method can be applied without actually accumulating the fund. The procedure is the same as though the fund were actually created.

Procedure: (*a*) Find the amount of the total depreciation of the asset by deducting the scrap value from the cost.

(*b*) Divide the total depreciation found in (*a*) by the amount of an ordinary annuity of 1 at the sinking fund interest rate, for the number of periods of the life of the asset, $s_{\overline{n}|i}$. This will give the periodic sum to be placed in the sinking fund.

(*c*) To the periodic sum found in (*b*), add a sum equal to the interest on the sinking fund for the period. This will give the periodic charge to depreciation, and the credit to reserve for depreciation.

Example

An asset costs \$1,000, and has a scrap value of \$100 at the end of 10 years. Determine the periodic depreciation charge by the sinking-fund method, on a 6% interest basis.

Formula

$$\frac{\text{Cost} - \text{Scrap}}{s_{\overline{n}|i}} = \text{Periodic deposit to sinking fund.}$$

Arithmetical Substitution

$$\frac{1{,}000 - 100}{\frac{(1.06)^{10} - 1}{.06}} = \$68.28.$$

The formula and substitution just shown give only the first periodic charge to depreciation. Each periodic charge thereafter is an amount equal to the sum of the first periodic payment and the interest on the accumulated depreciation reserve. Table II, on page 410, shows the periodic charges to depreciation and the credits to the reserve account for each of the 10 years.

The depreciation entries are independent of the fund entries. If a sinking fund were provided for the above example, the entries for the fund would be as shown in Table I:

TABLE I—ENTRIES TO THE SINKING FUND

Year	*Debit to Sinking Fund*	*Credit to Cash*	*Credit to Interest*	*Accumulation of Fund*
1	\$ 68 28	\$ 68.28	\$..........	\$ 68 28
2	72 38	68.28	4.10	140 66
3	76 72	68 28	8.44	217.38
4	81.32	68 28	13.04	298 70
5	86.20	68 28	17 92	384.90
6	91.37	68 28	23 09	476.27
7	96.86	68.28	28 58	573.13
8	102.67	68.28	34.39	675.80
9	108 83	68.28	40 55	784.63
10	115 37	68.29	47.08	900.00
	\$900.00	\$682.81	\$217.19	

TABLE II—DEPRECIATION ENTRIES BY THE SINKING-FUND METHOD OF DEPRECIATION

End of Year	*Depreciation Charge and Reserve Credit*	*Accumulated Depreciation Reserve*	*Asset Value*
			$1,000 00
1	$ 68 28	$ 68 28	931 72
2	72 38	140 66	859 34
3	76 72	217 38	782 62
4	81 32	298 70	701 30
5	86 20	384 90	615 10
6	91 37	476 27	523 73
7	96 86	573.13	426 87
8	102 67	675 80	324 20
9	108 83	784 63	215 37
10	115 37	900 00	100.00

Problems

1. Set up sinking fund depreciation tables for the following assets, using 5% as the sinking fund rate.

Asset	*Cost*	*Scrap*	*Life*
Office furniture .	$ 500	$ 125	8 years
Factory furniture	1,000	100	10 years
Machinery	15,000	3,000	8 years
Delivery equipment .	4,000	500	5 years

2.* The owner of an unimproved building site who is desirous of developing it so that it will produce an income, receives from a proposed lessee a proposition relative to the erection of a building at a cost of $100,000. In calculating the annual expenses, which are to be made the basis of rentals, the owner assumes a life of 50 years for the proposed building, and calculates by the straight-line method that the depreciation charge should be $2,000 per year. The prospective lessee contends that this depreciation charge is too large, that, as depreciation charged into expense does not represent actual expenditures, an amount of cash equal to the depreciation charge should be set aside annually and invested in interest-bearing securities, and the interest obtained annually reinvested. He demonstrates by calculation that an annual depreciation charge of $477.68 so handled will at an interest rate of 5% amount to $100,000 at the end of 50 years, and hence argues that this amount rather than $2,000 should be taken into consideration in determining the annual rental.

You are asked by the owner to give your opinion as to which of the two methods should be used, and why. Give your answer.

Annuity method of depreciation. The theory applied in this method is that the depreciation charge should include, in addition to the amount credited to the reserve, interest on the carrying value of the asset.

The investment in property is regarded, first, as the amount of scrap value which draws interest, and second, as an investment in

* C. P. A., Wisconsin.

an annuity to be reduced by equal periodic amounts. The interest on the scrap value plus the equal periodic reduction of the investment is the charge to depreciation, offset by a credit to interest computed on the diminishing value of the property, and a credit to the reserve account for the balance. This charge to depreciation is the same each period during the life of the property. The theory of an investment in an annuity is that the annuity is to be reduced by equal periodic payments, and as the credits to interest will decrease, the credits to the reserve must correspondingly increase.

Procedure: (*a*) Find the difference between the cost and the scrap value.

(*b*) Divide the difference found in (*a*) by the present value of an annuity of 1.

(*c*) Calculate the interest on the scrap value for one period at the given rate per cent.

(*d*) Determine the sum of (*b*) and (*c*). This sum will be the periodic charge to depreciation.

Example

Calculate by the annuity method the annual charge to depreciation for an asset valued at $1,000, with a scrap value of $100, which is to be written off in 10 years on a 6% basis.

Formula

$$\left[\frac{\text{Cost} - \text{Scrap}}{a_{\overline{n}|i}}\right] + (\text{Scrap} \times i) = \text{Periodic charge.}$$

Arithmetical Substitution

$$\left[\frac{1{,}000 - 100}{\dfrac{1 - \dfrac{1}{(1.06)^{10}}}{.06}}\right] + (100 \times .06) = \$128.28.$$

Solution, Part 1

$\$1{,}000 - \$100 = \$900$, sum to be depreciated

$(1.06)^{10} = 1.7908477$, compound amount of 1 for 10 periods at 6%

$1 \div 1.7908477 = .5583948$, present value of 1 for 10 periods at 6%

$1 - .5583948 = .4416052$, compound discount on 1 for 10 periods at 6%

$.4416052 \div .06 = 7.360087$, present value of an annuity of 1 for 10 years at 6%

$\$900 \div 7.360087 = \$122.28$, rent of the present value of annuity

Solution, Part 2

$100 × .06 = $6.00, interest on scrap value

Solution, Part 3

$122.28 + $6.00 = $128.28, periodic charge to depreciation

In the above example, the $900 represents the present value of the sum to be spread over the life of the asset, and the $100 represents the scrap value. In the following tables, the fifth column always contains the carrying value of the annuity, plus $100. The two tables are given to show the similarity between an annuity in which an investment was made and equal annual rents withdrawn, and the annuity method of depreciation.

TABLE OF REDUCTION OF AN ANNUITY

End of Period	*Rents Withdrawn*	*Credits to Interest*	*Amortization of Investment*	*Present Value of Annuity, Plus $100*
				$1,000 00
1	$ 128 28	$ 60 00	$ 68.28	931 72
2	128 28	55 90	72 38	859 34
3	128 28	51 56	76 72	782 62
4	128 28	46 96	81.32	701 50
5	128 28	42.08	86 20	615 10
6	128.28	36.91	91 37	523 73
7	128 28	31.42	96 86	426 87
8	128 28	25.61	102 67	324 20
9	128 28	19.45	108 83	215.37
10	128 29	12 92	115 37	100.00
	$1,282 81	$382 81	$900.00	

TABLE OF REDUCTION OF THE VALUE OF AN ASSET

End of Period	*Depreciation Charge*	*Credits to Interest*	*Credits to Reserve*	*Value of Asset*
				$1,000 00
1	$ 128.28	$ 60 00	$ 68 28	931 72
2	128 28	55 90	72 38	859 34
3	128 28	51.56	76.72	782 62
4	128 28	46.96	81 32	701 50
5	128 28	42 08	86 20	615 10
6	128 28	36 91	91 37	523 73
7	128 28	31.42	96 86	426 87
8	128.28	25 61	102 67	324 20
9	128.28	19 45	108 83	215 37
10	128 29	12.92	115 37	100.00
	$1,282.81	$382.81	$900.00	

Problem

The Acme Manufacturing Company, believing that the annuity method of depreciation is the correct one, desires that you construct tables for the following machines (one table for each):

Assets	*Cost*	*Scrap*	*Life*	*Interest Rate*
Lathes	$5,000	$ 500	10 years	5%
Milling machines...........	4,500	1,000	8 years	5%
Grinders...................	1,200	200	5 years	5%
Motors.	2,000	200	6 years	5%

Fixed-percentage-of-diminishing-value method. By this method a uniform rate on diminishing value gives the amount to be charged to depreciation each year. As the book value declines each year, the percentage of book value declines similarly.

The difficulty encountered in this method is that of finding the rate per cent to be used in the calculation of the charge.

Procedure: (*a*) Divide the scrap value by the cost.

(*b*) Extract the root, the index of which corresponds to the number of periods of depreciation to be taken on the life of the asset, of the quotient obtained in (*a*).

(*c*) Deduct from 1 the result obtained in (*b*), to find the rate per cent to be used.

(*d*) Multiply the net asset value or the carrying value of the asset at the beginning of each period by the rate found in (*c*), to obtain the depreciation charge for each period.

Example

What will be the depreciation charges for an asset valued at $1,000, with a scrap value of $100, which is to be written off in 10 years by the fixed-percentage-of-diminishing-value method?

The following are the formula and solution for the calculation of the rate:

Formula

$$1 - \sqrt[n]{\frac{\text{Scrap value}}{\text{Cost value}}} = r.$$

Arithmetical Substitution

$$1 - \sqrt[10]{\frac{100}{1{,}000}} = 20.5672\%.$$

Solution, Part 1

$$\begin{aligned}
100 \div 1{,}000 &= .1\\
\log .1 &= \bar{1}.000000\\
\text{Changed, } \bar{1}.000000 &= 9.000000 - 10\\
9.000000 - 10 \div 10 &= .900000 - 1\\
\text{Changed} &= \bar{1}.900000\\
\text{The antilog of } \bar{1}.900000 &= .794328\\
1 - .794328 &= .205672, \text{ or } 20.567\%
\end{aligned}$$

Solution, Part 2

$$\begin{aligned}
\$1{,}000 \times 20.567\% &= \$205.67, \text{ first depreciation charge}\\
\$1{,}000 - \$205.67 &= \$794.33, \text{ new asset value}\\
\$794.33 \times 20.567\% &= \$163.37, \text{ second depreciation charge}
\end{aligned}$$

This process is continued for each of the 10 years.

TABLE OF DEPRECIATION
(Rate, 20.567%)

Year	*Periodic Depreciation Charge*	*Accumulated Depreciation Reserve*	*Asset Value*
			$1,000 00
1	$205 67	$205 67	794 33
2	163.37	369 04	630 96
3	129.77	498 81	501 19
4	103 08	601 89	398 11
5	81 88	683 77	316 23
6	65 04	748 81	251 19
7	51 66	800 47	199 53
8	41 04	841 51	158 49
9	32 60	874 11	125 89
10	25 89	900 00	100 00

Problems

1. Construct a comparative columnar table showing the periodic depreciation charges computed by the straight-line, sum-of-digits, sinking-fund, annuity, and fixed-percentage-of-diminishing-value methods for an asset costing $10,000, and having a probable life of 10 years and a scrap value of $1,500. Use an interest rate of 4% per annum.

2. An asset costing $2,000 has a life of 5 years. It has no scrap value, but for the purposes of calculation, use $1. Money is worth 5%. Construct comparative columnar tables showing the carrying value and the annual depreciation charge computed by the straight-line, sum-of-digits, sinking-fund, annuity, and fixed-percentage-of-diminishing-value methods.

3. A businessman, having heard much about correct depreciation but understanding little of the methods of calculation used, calls on you to explain to him by means of comparison the five most important methods. As an illustration, use an asset costing $4,000, with a scrap value of $500 and a life of 5 years, and an interest rate of 6%.

Composite life. Often the depreciable assets of a business have wearing values (cost less scrap value) and terms of effective life that vary widely, yet it is desirable to ascertain the life of the plant as a whole, as when bonds secured by a mortgage on buildings and equipment are issued. The bonds should not be issued for a term of years exceeding the composite life of the plant; and, for a margin of safety, the term of the bonds should be considerably shorter than the composite life of the plant.

If interest *is not* a factor, the composite life is found by dividing the total wearing value by the total depreciation.

Unit	*Life*	*Cost*	*Scrap*	*Wearing Value*	*Annual Charge*
Bldg. (Brick)...........	50	$45,000	$5,000	$40,000	$ 800
Bldg. (Frame)..........	25	12,000	2,000	10,000	400
Heavy Machinery.......	25	30,000	5,000	25,000	1,000
Boiler.................	20	12,000	2,000	10,000	500
				$85,000	$2,700

85,000 ÷ 2,700 = 31.5, approximate years

If interest *is* a factor, as when depreciation is computed on the sinking fund basis, the annual charge for depreciation is the annual rent, the accumulation of which should equal the wearing value. Making use of the data from above and using 5% as the interest rate, we have:

Unit	*Life*	*Wearing Value*	*Annual Charge*
Bldg. (Brick)........	50	\$40,000	\$ 191.07
Bldg. (Frame)..........	25	10,000	209.53
Heavy Machinery.......	25	25,000	523.81
Boiler..................	20	10,000	302.43
		\$85,000	\$1,226.84

$$1{,}226.84 \div 85{,}000 = 1.4433\%$$

The per cent is low because the major asset has a 50-year life, and because over long periods the interest is also a large factor.

To find the composite life, let r denote the rate of depreciation and i the interest rate. Then,

$$n = \frac{\log\left(1 + \frac{i}{r}\right)}{\log(1 + i)}$$

For the foregoing problem

$$n = \frac{\log\left(1 + \frac{.05}{.014433}\right)}{\log 1.05}$$

$$= \frac{\log 4.464}{\log 1.05} = \frac{.649724}{.021189}$$

$$.649724 \div .021189 = 30.66 \text{ years.}$$

Problems

1. A company's plant consists of: (*a*) Buildings: cost, \$100,000; life, 40 years, scrap value, \$10,000. (*b*) Engine: cost, \$40,000; life, 25 years; scrap value. \$5,000. (*c*) Boiler: cost, \$12,000; life, 20 years; scrap value, \$2,000. (*d*) Electrical equipment: cost, \$7,500; life, 15 years; scrap value, \$1,500. Compute the charge for depreciation by the sinking fund method, on a 4% basis, and find the composite life of the plant.

2. Find the composite life of a plant consisting of the following:

Item	*Cost*	*Scrap Value*	*Life*
A	\$ 6,000	\$ 200	10 years
B	3,500	500	15 years
C	12,000	1,000	20 years
D	15,000	2,500	12 years

Interest at 5%.

Depreciation Problems from C. P. A. Examinations

1.* A manufacturing company has a factory building which cost, with its equipment, \$100,000. The company has set up a depreciation reserve of \$30,000.

* C. P. A., Michigan.

An appraisal made shows the replacement cost to be $160,000, and the depreciated sound value to be $130,000.

(*a*) Prepare the necessary entries to give effect to the appraisal figures.

(*b*) How would you treat the item of depreciation on the increased values, for the purpose of determining costs?

2.* A machine which cost $1,200 has been used for 5 years, and has depreciated annually 10%. The latter amount has been credited to the Reserve account.

(*a*) At the end of the first 5 years, the machine is traded for a new one which costs $1,500; an allowance of $300 is made on the old machine, the balance being paid in cash. Prepare the necessary entries to take care of this transaction.

(*b*) Assume that the machine is traded for one costing $1,700, and that an allowance of $700 is made, the balance being paid in cash. Prepare the necessary entries.

3.† A manufacturing plant, operating to the date of negotiations relative to its disposition, was acquired by a newly formed corporation, the price being based on the present sound values, which were stated as follows:

	Present Sound Value	Age
Machinery	$116,500	$4\frac{1}{2}$ years
	26,300	4 years
	217,300	$2\frac{1}{2}$ years
	16,750	2 years
	57,550	1 year
Equipment.	$ 13,300	6 years
	11,650	2 years
	27,660	1 year
Buildings: *A* .	$285,700	12 years
A .	15,000	$5\frac{1}{2}$ years
A .	16,600	1 year
B .	525,000	5 years

The estimated life of the machinery is 10 years from the date of original purchase; of the equipment, 15 years from the date of purchase; of buildings *A*, 30 years; and of building *B*, 45 years.

It is desired to set up the assets on the books at present reproductive values, with a corresponding depreciation reserve to bring the net book value to the amount of the "sound values" given above. Compute the "reproductive value" and the depreciation reserve, and give the future annual depreciation provision, all on the basis of a uniform rate each year until the book value is extinguished.

It may be assumed for the purposes of your answer that the assets will have no salvage value.

4.‡ The City Dairy Company bottles and distributes milk. Its sales average 40,000 lbs. per day. It operates three pasteurizers, each of which has a capacity of 2,500 lbs. per hour.

These machines cost $1,200 apiece, installed, and they have been in use for 3 years. At the time that they were installed, their life was estimated at 15

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* C. P. A., Michigan.

† American Institute Examination.

‡ C. P. A., Wisconsin.

years, and their salvage value at the end of that period at $50 each. Experience has shown that repair and maintenance charges on these machines will average 3% of their cost per year.

The Smith Dairy Machinery Company manufactures a new type of machine which is guaranteed to have a productive capacity of 12,000 lbs. per hour, and to save, in comparison with the old type, 80% of the cost of live steam and refrigeration used in pasteurization. The life of these machines is estimated at 10 years, and their salvage value at the end of that period at $100 each. The manufacturers offer to install them ready to operate at $8,500 each, and to remove the old machines and make an allowance for their estimated scrap value. Their guarantee also provides for replacement of broken or defective parts, and for complete maintenance to keep the machines in good working order for 1 year. Subsequent repair and maintenance charges may be assumed to average 5% annually of the machines' original cost.

You are called upon to make a special examination of the accounts, with a view to determining whether it would be advantageous from a profit and loss standpoint to install the new type of machine. Your examination discloses that the present cost of live steam and refrigeration used in pasteurization averages 44¢ per 1,000 lbs. of milk.

Required: (*a*) Assuming future production to average 5% increase over the sales given above, state the conclusions that you would report to your client, and show your method of arriving at them.

(*b*) Assuming that the manufacturer's proposition has been accepted, draft the entries which you consider should be made to record the changes.

5.* The Plant and Equipment and Reserve for Depreciation accounts, presented below, represent the transactions of the A. Company for the year 1944, as recorded by the bookkeeper, from January 1 of that year.

PLANT AND EQUIPMENT

1944			1944		
Jan. 1	Balance	$500,000	Jan. 31	Screw-cutting lathe	$ 150
17	Planer	2,000	Apr. 17	Steam engine	300
Mar. 21	Bolt machine	1,250	Sept. 30	Steel and lumber	400
Apr. 16	Crane	3,000	Dec. 31	Balance	524,650
May 3	Electrical equipment for crane	1,200			
27	Roof of machine shop	3,500			
June 3	Lathe belting	750			
Aug. 20	Wm. Smith, Contractor	10,000			
Dec. 31	Machine shop	3,800			
		$525,500			$525,500

RESERVE FOR DEPRECIATION—PLANT AND EQUIPMENT

1944			1944		
Dec. 31	Balance	$175,000	Jan. 1	Balance	$125,000
			Dec. 31	Depreciation at 10% per annum	50,000
		$175,000			$175,000

* Adapted from American Institute Examination.

The following is a description of the transactions; you are required to make any entries that you deem necessary to correct the accounts, giving reasons therefor, and setting up corrected accounts.

The balances at the beginning are assumed to be correct.

Planer, $2,000, is a standard machine, purchased new.

Bolt machine was made in company's own shop. The $1,250 represents cost of castings, $500, and direct labor, $750. The machine shop pay roll was $20,000 ($15,000 direct, and $5,000 indirect) during the year; castings and parts purchased were $17,000; general supplies were $4,000; rent was $2,500; light heat, and power were $3,500.

Crane and equipment, $4,200, are standard machinery, purchased new.

Roof of machine shop was destroyed by weight of snow during the winter.

Belting for all equipment, amounting to $25,000, was charged to plant and equipment when the plant was opened, and has not since been depreciated.

William Smith is engaged in erecting an addition to the plant buildings. $10,000 is the first payment on the uncompleted work.

Machine shop, $3,800, represents the cost of making tools, setting machines, and installing new machinery, as follows:

Tool making	$1,000
Setting machines for special work	1,800
Installing planer	300
Installing bolt machine	200
Installing crane	500
	$3,800

Screw-cutting lathe—cost, 1937, $2,000.

Steam engine—cost, 1934, $15,000.

Steel and lumber, $400, represents salvage from machine shop roof.

Prior to December 31, 1943, a separate account was kept for land and buildings.

Ten per cent per annum depreciation on plant and equipment has been written off.

6.* A machine costing $81 is estimated to have a life of 4 years and a residual value of $16. Prepare a statement showing the annual charge for depreciation computed by each of the following methods: (*a*) straight-line; (*b*) constant-percentage-of-diminishing-value; (*c*) annuity. (For convenience in arithmetical calculation, assume the rate of interest to be 10%.)

Depletion. The provision for the extinction of wasting assets, such as mines, timber lands, or gravel pits, is called a provision for depletion.

We shall deal with two classes of problems; namely, the determination of the amount of depletion each year, and the capitalization of the wasting asset.

Calculation of depletion. The amount of depletion usually stands in the same proportion to the total cost of the wasting asset as the units of product removed stand to the total units of product that it was estimated the asset would produce when new; but if the

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* American Institute Examination.

property is leased and it is not possible to remove all the product before the lease expires, it is plain that depletion should be based on the quantity to be extracted during the period of the lease.

Problems

1. The estimated recoverable tonnage in a coal mine was placed at 2,214,363 tons. The value of "Coal Lands" was $96,443.62. Compute the depletion charge per ton of coal mined.

2. A tract of timber was valued at $25,965.86, and its footage was estimated at 17,228,000. The following year the timber cut was 5,184,336 feet. Compute the rate of depletion per thousand feet, and the depletion charge for the year.

3. A mining property was valued at $50,000, and the estimated recoverable tonnage placed at 1,000,000 tons.

During the next 7 years, tonnage was removed as follows:

First year	50,000 tons
Second year	60,000 tons
Third year	70,000 tons
Fourth year	80,000 tons
Fifth year	80,000 tons
Sixth year	80,000 tons
Seventh year	80,000 tons

Prospecting and development toward the close of the seventh year cost $25,000, and resulted in an estimated recoverable tonnage of 2,000,000 tons. Appreciation due to discovery was placed on the books at $60,000.

The tonnage removed during the eighth and ninth years was 100,000 tons and 120,000 tons, respectively.

Calculate the annual charges for depletion.

[Suggestion: Set up two accounts, Mining Property (Cost), and Mining Property (Discovery Value), and credit these accounts with the proper depletion charges each year. This problem is illustrative of the complications that arise when there is more than one valuation on a particular property.]

Capitalized cost. Capitalization is the cost of an indefinite number of renewals of anything. The value is found as a perpetuity, page 364. Capitalized cost is the cost of the renewals plus the original cost. When an endowment fund provides for the periodic replacement of a useful memorial, such as a building, a bridge, and so forth, the amount of the fund is the capitalized cost of the memorial.

Procedure: (*a*) Divide the cost of the asset by the given rate per cent expressed decimally, to find the capitalization.

(*b*) Divide the capitalization found in (*a*) by the amount of an ordinary annuity of 1 at the given rate per cent and for the time, expressed in periods, which represents the life of the asset. (This will give the amount of an endowment fund necessary to replace the asset periodically at the end of each term of years.)

(c) Add the cost of the asset to the amount of the endowment fund found in (b). The sum of these two items is the amount necessary to provide for the first cost and for the replacement fund.

Example

It is desired to set aside a fund which will provide for the erection of a memorial costing $2,000, and for the replacement of the memorial at the end of each 5 years. Money is worth 6%. Find the amount of the fund.

Formula	*Arithmetical Substitution*
$\dfrac{\dfrac{\text{Cost}}{i}}{s_{\overline{n}\vert i}} + \text{Cost} = \text{Capitalized cost.}$	$\dfrac{\dfrac{2{,}000}{.06}}{\dfrac{(1.06)^5 - 1}{.06}} + 2{,}000 = \$7{,}913.21.$

Solution

2,000 ÷ .06 = 33,333.33, capitalization of asset
$(1.06)^5$ = 1.3382256, compound amount of 1 at 6% for 5 years
1.3382256 − 1 = .3382256, compound interest on 1 at 6% for 5 years
.3382256 ÷ .06 = 5.637093, amount of ordinary annuity of 1 at 6% for 5 years
33,333.33 ÷ 5.637093 = 5,913.21, sum available for investment at compound interest
$5,913.21 + $2,000 = $7,913.21, capitalized cost

Verification

$2,000 = present cost of memorial
$7,913.21 − $2,000 = $5,913.21, sum available for investment at compound interest
$5,913.21 × 1.3382256 = $7,913.21, total fund 5 years hence

Problems

1. The cost of the Davis Memorial Building was $150,000. It is estimated that the building will last 50 years. Money is worth 4%. Calculate the capitalized cost of the building.

2. A philanthropist desires to provide a fund for the erection of a college building costing $200,000, and for the replacement of the building at the end of each 50 years. Money is worth 4%. Calculate the amount of the fund, assuming that repairs and renewals necessary during the 50-year periods are not to be paid for out of the fund.

Perpetuity providing for ordinary annual expenses and for replacement of asset. Many endowments provide for ordinary annual expenses as well as for the replacement of the asset.

Procedure: (a) Divide the cost of the asset by the given rate per cent expressed decimally, to find the capitalization.

(b) Divide the capitalization of the asset by the amount of an ordinary annuity of 1 at the given rate per cent for a number of

periods, expressed in years, equal to the life of the asset. This will give the replacement fund.

(*c*) Divide the annual upkeep by the given rate per cent, to find the capitalization of the upkeep fund.

(*d*) Add the original cost of the asset, the replacement fund, and the upkeep fund to find the total fund necessary.

Example

The Rutledge Home cost $75,000. The annual expenses of running the institution, including repairs and upkeep, are estimated at $5,000. If the life of the building is 50 years, and money is worth 4%, what is the amount of the fund necessary to provide for the perpetuity of the home?

Formula

$$\frac{\frac{\text{Cost}}{i}}{s_{\overline{n}|i}} + \text{Cost} + \frac{\text{Upkeep}}{i} = \text{Sum.}$$

Arithmetical Substitution

$$\frac{\frac{75{,}000}{.04}}{\frac{(1.04)^{50} - 1}{.04}} + 75{,}000 + \frac{5{,}000}{.04} = \$212{,}281.63.$$

Solution, Part 1

$75,000 ÷ .04 = $1,875,000, capitalization
$1,875,000 ÷ $152 66708 = $12,281.63, building replacement fund

Solution, Part 2

$75,000 = cost of building

Solution, Part 3

$5,000 ÷ .04 = $125,000, upkeep fund

Summary

Building fund	$ 12,281.63
Cost of building	75,000 00
Upkeep fund	125,000 00
Total fund necessary	$212,281 63

Problems

1. The original cost of a public library is $100,000, the estimated life of the building is 50 years, and the annual cost of upkeep is $12,000. Money is worth 4%. Calculate the amount of the fund necessary to provide for the perpetuity of the library.

2. The cost of a memorial is $55,000, its life is 25 years, and its upkeep is $2,400 a year. Money is worth 4%. Calculate the amount of the fund necessary to provide for building the memorial, keeping it in repair, and replacing it at the end of each 25-year period.

3. A charitable institution is to be built and maintained by a trust committee. It is estimated that the building will cost $100,000, and that its life will be 60 years. The estimated annual income and costs are as follows:

Income:	
Donations	$4,000
Expenses:	
Miscellaneous service charges	1,000
Heat	2,000
Matron	1,800
Help	3,000
Food	5,000
Medical attention	1,200
Incidentals	1,000

Find the value of the endowment at 5%, interest convertible annually.

Capitalization of a wasting asset. Investment in a wasting asset such as a mine or timberlands should yield not only interest on the investment, but additional income to provide funds for the restoration of the capital originally invested. A sinking fund, called a *redemption fund,* is created for the purpose of restoring the original capital. The investment rate will usually be higher than the rate which can be earned on the redemption fund.

To find the fair market value at which a wasting asset may be capitalized, divide the estimated annual average income by the sum of: (*a*) the rent of an ordinary annuity of 1 at the redemption fund rate for the length of time estimated to deplete the asset (this to provide the annual sinking fund payment to restore the investment), and (*b*) a fair annual rate of income (this to provide a return to the investor).

Example

A mine produces an average annual operating income of $10,000. At the present rate of depletion, it is estimated that the mine will last 8 years. If the sinking fund will earn 4%, and the owners are to receive a dividend of 6%. what should be the capitalized value?

Formula

$$\frac{\text{Operating profit}}{\dfrac{1}{s_{\overline{n}|i}} + \text{Dividend rate}} = \text{Capitalized value}$$

Arithmetical Substitution

$$\frac{10{,}000}{\dfrac{1}{\dfrac{(1.04)^8 - 1}{.04}} + .06} = \$59{,}337.39$$

Solution

$(1.04)^8 = 1.3685691$, compound amount of 1 for 8 periods at 4%

$1.3685691 - 1 = .3685691$, compound interest on 1 for 8 periods at 4%

$.3685691 \div .04 = 9.214226$, amount of annuity of 1 for 8 periods at 4%

$1 \div 9.214226 = .1085278$, the rent of an annuity that will amount to 1

$.1085278 + .06 = .1685278$, rent of annuity plus the dividend rate

$\$10,000 \div .1685278 = \$59,337.39$, capitalized value

TABLE OF CAPITALIZATION OF WASTING ASSET

Years	*Annual Income*	*6% on Capitalized Value*	*Sinking Fund Payments*	*Interest on Sinking Fund*	*Sinking Fund Accumulations*
1	$10,000	$3,560 24	$6,439.76	$............	$ 6,439 76
2	10,000	3,560 24	6,439 76	257 59	13,137 11
3	10,000	3,560 24	6,439 76	525 48	20,102 35
4	10,000	3,560 24	6,439 76	804 09	27,346 20
5	10,000	3,560 24	6,439 76	1,093 85	34,879 81
6	10,000	3,560 24	6,439 76	1,395 19	42,714 76
7	10,000	3,560.24	6,439 76	1,708 59	50,863 11
8	10,000	3,560.24	6,439 76	2,034 52	59,337.39

Problems

1. Find the capitalized value of a coal mine which will produce a net income of $20,000 a year for 30 years; the annual income rate is 6%, and a sinking fund is to be accumulated at 4%.

2. A tract of timber will yield an annual revenue of $20,000 for 20 years. If annual dividends are declared at 5%, and payments are made annually into a sinking fund which bears 4% interest, what is the value of the timber rights?

3. A gravel pit is estimated to contain 3,500,000 cubic yards of gravel. This pit is leased at a royalty of 10¢ per cubic yard of gravel extracted, and the average annual output is 150,000 cubic yards. If a 6% dividend is paid on the stock, and a fund equal to the capital stock is accumulated at 4%, what should be the capitalized value of the property?

Review Problems

1. A machine that cost $1,000 is estimated to have a life of 10 years and a scrap value of $200. Compute the annual depreciation charge by:

(*a*) The straight line method.
(*b*) The fixed percentage of diminishing value method.
(*c*) The sinking fund method, using 6% effective interest
(*d*) The annuity method, using 6% effective interest.

2. A mine with a net annual yield of $75,000 will be exhausted in 15 years at the present rate of output. What is the mine worth, on a 5% basis?

3. A certain make of bench drill costs \$17.50 and lasts 3 years. How much can be paid for a better grade of drill that will last 6 years, money being worth 4%? (HINT: Capitalized costs must be equal. Solve for x.)

4. A roof made of one material will cost \$300 and last for 20 years. If made of another type of material, it will last for the life of the building, which is estimated to be 75 years. How much can one afford to pay for the permanent type of roof if money is worth 5%?

5. Calculate the fixed percentage to be written off each year for an asset costing \$2,400, estimated life 6 years and scrap value \$400.

CHAPTER 35

Building and Loan Associations

Control. Building and loan associations are organized under state laws, and in most cases are under the direct supervision of the state banking department. The banking department requires semiannual or annual reports, and the associations are subject to special examinations from time to time by the state bank examiners. In addition to this, annual audits are usually made by committees of stockholders, and in a great many cases independent audits are made by certified public accountants.

Classes of stock. The amount of capital stock of a building and loan association is fixed by the charter, and the minimum capital and the par value per share are stated at the beginning of the charter. The classes of stock issued are installment stock and fully-paid stock.

Installment stock. Ownership of this class of stock is generally evidenced by a pass book, in which the weekly or monthly payments are recorded. The monthly payments are usually $1 for each share with a maturity value of $200, or 50¢ for each share with a maturity value of $100. In some organizations, if it is desired to mature the shares in a shorter time, double payments may be made. Shares of this class participate in all the earnings of the association.

When the monthly installments, called dues, and the profits or dividends credited to the stockholder equal the face value of the shares, the shares are matured or paid-up. The amount may then be withdrawn, or fully-paid shares may be issued.

If the stockholder, also called a member, has borrowed from the association, the maturing of his stock effects a reduction in his indebtedness.

Fully-paid stock. Ownership of this class of stock is evidenced by a stock certificate, and the stockholder usually receives a given rate of interest, although in some cases stockholders participate in earnings in the same manner as holders of installment stock.

Withdrawal of funds. In most cases, funds deposited in an association must be left for at least six months. After that time any member may withdraw all or a part of his funds, together with the dividends credited and not already paid. In the event of withdrawal before the maturity of the shares, some associations retain

a membership or withdrawal fee of 2% of the par value of each share, and issue membership certificates which are transferable upon the books of the association. The member may retain this certificate, or may assign it to someone else, as he sees fit. When another account is opened in the association, either by the member or by anyone else holding his certificate of membership, credit is given for the amount which the certificate represents.

Other associations permit the withdrawal of the face amount of the deposits, and allow the holder interest for the equated time. The rate of interest is fixed by the association, and is usually lower than the per cent earned by the shares. This difference in rates results in a profit to the association. Such profit on withdrawals is added to the other profits, and distributed to the shares remaining in the association.

Plans of organization. There are three principal plans upon which building and loan associations are organized: the terminating plan; the serial plan; and the permanent or perpetual plan, also called the Dayton or Ohio plan.

Terminating plan. The life of the association is limited under the terminating plan to any number of years that may be agreed upon. When the specified number of years elapses, the association goes out of existence, or a new one may be formed for another period of time. On becoming a member subsequent to the date of issue of the stock, the purchaser pays the book value of the share or shares purchased. The book value consists of back dues, plus dividends that have accumulated to the credit of the stock since the date of original issue.

The amount of the monthly payments necessary to retire the stock is determined in the same manner as is the rent of the present worth of an annuity.

Determination of the amount to be paid monthly under the terminating plan.

Procedure: (*a*) Compute the interest rate per period by dividing the annual rate by the number of times that conversion takes place.

(*b*) Compute the number of periods by multiplying the time in years by the number of times that the interest is converted annually.

(*c*) Compute the present value of an annuity of 1, using the periodic rate found in (*a*) and the number of periods found in (*b*).

(*d*) Divide the par value of one share by the present value of an annuity of 1 found in (*c*).

Example

What should be the monthly payment per share, each share having a par

value of \$100, if it is desired to mature the shares in 5 years, money being worth 6%?

Solution

$$\frac{P}{a_{\overline{n}|i}} = \text{Monthly payment}$$

$$a_{\overline{60}|\frac{1}{2}\%} = 51.7256$$

$$100 \div 51.7256 = 1.933$$

$$\therefore \$1.93 = \text{Monthly payment.}$$

By this plan, the association would be dissolved at the end of the 5-year period.

Serial plan. Under this plan, stock is issued at specified dates. Each issue constitutes a new series, and shares in the profits in proportion to the length of time that the series is outstanding. The distribution of profits under this plan is similar to the distribution in a partnership using the average investment method. New members may pay all back payments to the beginning of the current series plus a charge for interest on these payments, or they may wait for a new series to begin; a new series may begin annually, semiannually, quarterly, or monthly, depending on the demand for shares. As soon as a new series is opened, issuance of shares in the previous series is stopped.

Distribution of profits. When several series of stock, maturing at as many different dates, have been issued, the distribution of profits is a complicated matter. Of the many methods of determining the proper profit distribution, the most familiar are the partnership method and Dexter's Method, commonly called "Dexter's Rule." The purpose of these methods is to secure an equitable distribution of profits among the members of the association, upon the basis of the amounts that they have contributed against the face value of the shares registered in their names, and upon the basis of the time that each dollar paid by the association members has been in the possession of the association.

Partnership method. The name indicates the substance of this method, and an example and solution are given to show its application.

Example

The Alpha Building and Loan Association issued six series of shares, as follows:

Series	*Date*	*Number of Shares*
1	Jan. 1, 1942	500
2	July 1, 1942	500
3	Jan. 1, 1943	400
4	July 1, 1943	300
5	Jan. 1, 1944	400
6	July 1, 1944	400

The dues in each series were $1 a share, payable monthly. The net profits from interest, fines, and so forth, less the operating expenses for the half-year ended Dec. 31, 1944, were $965.22, and the undivided profits on July 1, 1944, were $4,272.78.

The status of the shares on July 1, 1944, was as follows:

Series	*Date of Issue*	*Number of Shares*	*Paid per Share*	*Profit per Share*	*Value per Share*
1	Jan. 1, 1942	500	$30	$4 12	$34 12
2	July 1, 1942	500	24	2.66	26 66
3	Jan. 1, 1943	400	18	1.51	19 51
4	July 1, 1943	300	12	.69	12 69
5	Jan. 1, 1944	400	6	.19	6 19

Distribute the profits for the half-year ending December 31, 1944, by the partnership method.

Solution

On December 31, 1944, the amount paid on each share of each series was as follows:

Series	*Paid per Share*	*Series*	*Paid per Share*
1	$36	4	$18
2	30	5	12
3	24	6	6

Dues of $1 have been paid at the beginning of the month on each of the 500 shares in the first series for 36 months. The average time is found to be $18\frac{1}{2}$ months. That is, $1 was invested for 36 months; $1 for 35 months; $1 for 34 months; and so forth. The average, $18\frac{1}{2}$ months, is the sum of the first term and the last term, divided by 2; thus, $\frac{36 + 1}{2} = 18\frac{1}{2}$. Solving for series 2, 3, 4, 5, and 6, we have $15\frac{1}{2}$, $12\frac{1}{2}$, $9\frac{1}{2}$, $6\frac{1}{2}$, and $3\frac{1}{2}$, respectively, as the average time for each series.

As there are 500 shares in the first series, and each share has paid $36, the capital of the first series is $18,000. $18,000 for $18\frac{1}{2}$ months equals $333,000 for 1 month. Solving for each series, we have the following:

Series		*Dollars for 1 Month*
1	$36 × $18\frac{1}{2}$ months × 500 shares	$333,000
2	30 × $15\frac{1}{2}$ " × 500 "	232,500
3	24 × $12\frac{1}{2}$ " × 400 "	120,000
4	18 × $9\frac{1}{2}$ " × 300 "	51,300
5	12 × $6\frac{1}{2}$ " × 400 "	31,200
6	6 × $3\frac{1}{2}$ " × 400 "	8,400
		$776,400

Undivided profits, July 1, 1944	$ 4,272 78
Earnings, July 1, 1944, to Dec. 31, 1944	965 22
Profits available for distribution	$ 5,238.00

The profits available for distribution, $5,238, are prorated among the series in the proportion that the equated capital of each series bears to the total equated capital, as follows:

Series						
1	$\frac{3,330}{7,764}$ of \$5,238	= \$2,246 59	for 500	shares,	or \$4.49	per share
2	$\frac{2,325}{7,764}$ " "	= 1,568 57	" 500	"	" 3.14	" "
3	$\frac{1,200}{7,764}$ " "	= 809 58	" 400	"	" 2.02	" "
4	$\frac{513}{7,764}$ " "	= 346.10	" 300	"	" 1.15	" "
5	$\frac{312}{7,764}$ " "	= 210 49	" 400	"	" .53	" "
6	$\frac{84}{7,764}$ " "	= 56 67	" 400	"	" .14	" "
		\$5,238 00				

Dexter's rule for distribution of profits. This method is a modification of the partnership method. Its principles are illustrated in the following solution, which is based on the example given under the partnership method.

Solution

To the capital of each series on July 1, 1944, as shown by the column "Value per Share" in the previous tabulation, should be added the contribution of \$1 per share for each of the six months of the current half-year, computed by the average method.

\$1 paid	July 1	= \$1 for 6 months,	or \$ 6	for 1	month
1 "	Aug. 1	= 1 " 5 "	" 5	" 1	"
1 "	Sept. 1	= 1 " 4 "	" 4	" 1	"
1 "	Oct. 1	= 1 " 3 "	" 3	" 1	"
1 "	Nov. 1	= 1 " 2 "	" 2	" 1	"
1 "	Dec. 1	= 1 " 1 month,	" 1	" 1	"
			\$21	" 1	"

\$21 ÷ 6 = \$3.50, average for 6 months

Add \$3.50 to the value of each share, and multiply by the number of shares, to find the capital value of the series.

Series	*Shares*	*Dollars per Share*	*Capital per Series*
1	500	\$37 62	\$18,810 00
2	500	30 16	15,080 00
3	400	23 01	9,204 00
4	300	16 19	4,857 00
5	400	9 69	3,876 00
6	400	3 50	1,400 00
Total capital			\$53,227 00

The previous distributions of profits are unchanged. To these are added the profits of the current period, computed on the basis of the present earning capitals; in other words, the capital \$53,227 earned a profit of \$965.22, or 1.8134%.

Hence, the profit for each series and for each share in a series is calculated as follows:

Series	*Capital*	*Rate*	*Profit per Series*	*Shares*	*Profit per Share for Last Period*
1	$18,810	.018134	$341.10	500	$0.68
2	15,080	.018134	273.46	500	.55
3	9,204	.018134	166 91	400	.42
4	4,857	.018134	88 08	300	.29
5	3,876	.018134	70 29	400	.18
6	1,400	.018134	25 38	400	.06
			$965 22		

Adding the profit per share for the last period to the value of the share as of July 1, 1944, and including the monthly payments for the period, gives the value per share as of December 31, 1944.

COMPARISON OF PARTNERSHIP PLAN AND DEXTER'S RULE

	Partnership Plan			Dexter's Rule		
Series	*Profit per Share to July 1, 1944*	*Profit per Share to Dec. 31, 1944*	*Profit per Share for Last 6 Mos.*	*Profit per Share for Last 6 Mos.*	*Difference Inc.*	*Difference Dec.*
1	$4 12	$4 49	$0 37	$0.68	$0.31	
2	2 66	3.14	.48	.55	.07	
3	1 51	2.02	.51	.42		$0.11
4	.69	1.15	.46	.29		.17
5	.19	.53	.34	.18		.16
6	.00	.14	.14	.06		.08

The above comparison shows that the partnership plan favors the newer series at the expense of the old. This is unjust, because it was really the old series that produced the profits. Dexter's Rule is thus a more equitable method of calculating the distribution of profits in a building and loan association using the serial plan.

Problems

1. A building and loan association issued a new series each year, as follows:

Series	*Date of Issue*	*Number of Shares*
1	Jan. 1, 1941	300
2	Jan. 1, 1942	400
3	Jan. 1, 1943	500
4	Jan. 1, 1944	300

The dues were $1 per month per share. The profits to the end of the fourth year were $4,800. Find the value of one share in each series at the end of the fourth year, using the partnership method.

2. The first series of a certain building and loan association is 3 years old and has 1,000 shares; the second series is 2 years old and has 500 shares; and the third series is 1 year old and has 400 shares. The net assets are $60,650.00 Payments were $1 per month per share.

By the partnership method, compute: (*a*) net profits; (*b*) profit for 1 share in each series; (*c*) value of 1 share in each series.

3. By Dexter's Rule, find the value as of June 30, 1943, of 1 share in each series:

STATEMENT, DECEMBER 31, 1942

Series	*Date Issued*	*Number of Shares*	*Paid per Share*	*Profit per Share*	*Value per Share*
1	Jan. 1, 1941	500	$24	$3 32	$27 32
2	July 1, 1941	400	18	1 89	19 89
3	Jan. 1, 1942	300	12	.86	12 86
4	July 1, 1942	400	6	23	6 23

The fifth series was issued January 1, 1943, and comprised 300 shares. The dues in each series were $1 per month per share. The profits for the six months ended June 30, 1943, after all expenses had been deducted, amounted to $648.75.

4. The sixth of the above series was issued July 1, 1943, and comprised 400 shares. The net profits for the six months ended December 31, 1943, were $698.75. Find the value of 1 share in each series as of December 31, 1943.

Withdrawal value. If 20 shares of the third series in the example on page 427 are withdrawn on October 1, 1944, what is their withdrawal value, assuming that the association allows 5% interest on shares withdrawn before maturity?

Each share of the third series has a paid up value on October 1, 1944, of $21; hence the 20 shares are worth $420. To this amount add 5% interest for the equated time, 11 months, $\left(\frac{21 + 1}{2}\right)$, or $19.25; this gives a withdrawal value of $439.25. The book value of each share of this series is $22.51 ($21 in each payment, plus $1.51 profits as of July 1, 1944). Twenty shares at $22.51 gives a book value of $450.20. The difference, $10.95, is the profit on the transaction, and makes the divisible profits for the half-year $976.17 ($965.22 + $10.95).

Problems

(*Based on example, page* 427.)

1. Find the withdrawal value of 10 shares of the second series, withdrawn December 31, 1944, interest allowed at 4%.

2. Find the withdrawal value of 20 shares of the fourth series, withdrawn November 1, 1944, interest allowed at 5%.

3. Find the withdrawal value of 10 shares of the third series, withdrawn July 1, 1944, assuming that the association permits withdrawals at book value, but retains a membership fee of 2% of the par value of each share.

Dayton or Ohio plan. The plan most commonly used by building and loan associations is the Dayton or Ohio plan. Use of this plan eliminates the uncertainty as to the time of the maturity of a loan, thus giving the borrower a definite contract. Under

other plans a successful association would soon mature its stock, while an unsuccessful one would greatly prolong the time during which the borrower would have to continue his interest payments. Furthermore, under the Dayton plan the borrower does not have to own stock. Each payment that the borrower makes does two things: first, it pays the interest; second, it reduces the principal. The same idea is applied in the Federal Farm Loan Act, and is an application of the subject "Payment of debt by installments," discussed on page 339. The monthly payment is found by the formula used under the terminating plan.

Problems

1. Find the monthly payment which will cover both principal and interest of a $4,000 loan at 5% for 10 years.

2. Construct a schedule for the first 2 years.

3. Construct a schedule for the last 2 years, showing the final amortization of the debt.

To find the time required for stock to mature (rate of interest given). Such a problem is simply that of finding the time that it takes an annuity of annual rents payable in twelve monthly installments to accumulate to a certain amount.

Procedure: (*a*) Divide the maturity value by the number of dollars in the periodic payment.

(*b*) Multiply the quotient found in (*a*) by the periodic rate per cent.

(*c*) Determine the logarithm of 1 plus the product found in (*b*).

(*d*) Determine the logarithm of 1 plus the periodic rate per cent.

(*e*) Divide the logarithm found in (*c*) by the logarithm found in (*d*).

Example

The Washington Building and Loan Association yields the investor a nominal rate of 7%, convertible monthly. What is the time required for payments of $1 a month to mature $100?

Formula

$$\frac{\log\left[1 + \left(\dfrac{\text{Maturity value}}{\text{Periodic payment}} \times i\right)\right]}{\log(1+i)} = \text{Term.}$$

Arithmetical Substitution

$$\frac{\log\left[1 + \left(\dfrac{100}{1} \times .0058333\right)\right]}{\log 1.0058333} = 79 \text{ periods.}$$

Solution

Dividing:	$100 \div 1 = 100$	(*1*)
Multiplying:	$100 \times .0058333 = .58333$	(*2*)
Adding 1:	$1 + .58333 = 1.58333$	(*3*)
log:	$1.58333 = 0.199572$	(*4*)
log:	$1.0058333 = 0.002526$	(*5*)
Dividing:	log $0.199572 \div$ log $0.002526 = 79$ (approx.)	(*6*)

Problems

1. If the nominal rate of interest in the above example had been 5%, what would have been the time of maturity?

2. If the monthly payment is 50¢ and the nominal rate is 5%, convertible monthly, what time is required to mature a $100 share?

3. If payments of 50¢ a month on a $100 share earn for the investor an effective rate of 6%, convertible annually, in what time will the stock mature?

To find the effective rate of interest on money invested in installment shares. This is a problem similar to that of finding the effective rate earned on an annuity.

Example

On July 1, 1937, the Zenith Building and Loan Association issued $100 par value stock on which monthly payments of $1 per share were to be made. The stock matured on January 1, 1944, by the association's accepting 2¢ on the 80th payment of $1. What was the effective rate of interest earned?

To find the effective rate, it is necessary to use estimated rates, as explained under the heading "Computation of the rate of an annuity," page 343. The formula for the first estimated rate is as follows:

Formula	*Arithmetical Substitution*
$P\left(\frac{(1+i)^n - 1}{i}\right) = \text{Amount.}$	$1\left(\frac{(1.005666)^{79} - 1}{.005666}\right) = \$99.28.$

Solution

First trial rate, 6.8%:

$$.068 \div 12 = .005\tfrac{2}{3}\text{, monthly rate}$$
$$1.005\tfrac{2}{3} \text{ to 79th power (by logs)} = 1.5625808\text{, compound amount}$$
$$1.5625808 - 1 = .5625808\text{, compound interest}$$
$$.5625808 \div .005\tfrac{2}{3} = \$99.28\text{, amount of annuity}$$

The first trial rate is found to be too small.

Second trial rate, 7.2%:

$$.072 \div 12 = .006\text{, monthly rate}$$
$$1.006 \text{ to 79th power (by logs)} = 1.6041402\text{, compound amount}$$
$$1.6041402 - 1 = .6041402\text{, compound interest}$$
$$.6041402 \div .006 = \$100.69\text{, amount of annuity}$$

The second trial rate is found to be too large.

Interpolation

Value at 7.2%	$100 69
Value at 6.8%	99 28
Difference of .4%	$ 1.41

$1.41 ÷ 4 = .35, the difference represented by .1%

$100 − .02, excess of last payment	$99 98
Value of 79 payments at unknown rate	99 98
Value of 79 payments at 6.8%	99 28
Excess over rate of 6.8%	$ .70
Difference in value represented by difference of .1%	.35

.70 ÷ .35 = 2, or .2% to be added to 6.8%, or 7%

Problems

1. If, in the foregoing example, the stock had been matured by the association's acceptance of the full amount of the 70th payment, what would have been the effective rate earned?

2. Payments of 50¢ a month mature $100 in 10 years, 4 months, without its being necessary for any part of the 125th payment to be made. What is the effective rate of interest earned?

Classified Problems on Building and Loan Associations

Distribution of profits to shareholders.

1. Five shares of stock in a building and loan association had a book value of $215.80 at the beginning of a 6 months' period. The dues of $5 per month for the next 6 months, payable in advance, were paid when due. What is the average book value for the period that should be used in the distribution of profits to these 5 shares?

2. *B* subscribed for 20 shares in a building and loan association, and because his subscription was made at a time between dividend dates, he had to pay $20 in dues each month for 4 months. What was the book value of his payments?

3. Twenty shares of stock had a book value of $800 at the beginning of a 6 months' period. The shareholder became delinquent for 3 months. At the beginning of the fourth month he paid $80, and then paid $20 each month for the next 2 months. What was the average book value of his 20 shares? If his delinquency had been covered by fines, and he had therefore been allowed full participation in profits, what would have been the book value?

4. Ten shares of stock in the X. Building and Loan Association had a book value of $365.80 on July 1, 1943. Dues of $1 per month per share were paid for the next 6 months. On December 31, 1943, the average book value of holdings in the association was $126,178.36. The association reported a net gain for the 6 months amounting to $4,116.80. Find: (*a*) the rate per cent earned; (*b*) the dividend on the 10 shares, December 31, 1943; and (*c*) the book value of the 10 shares, December 31, 1943.

5. Adams paid $70 in advance for 1 share of paid-up stock in the Ames Building and Loan Association. The maturity value of the shares was placed at $100. The per cents of earnings for the 5 succeeding semiannual periods were 4.6%, 3.9%, 4.2%, 5.1%, and 4.9%, respectively. What was the book value of Adams' share at the beginning of the sixth period?

6. The Garfield Building and Loan Association issued shares at the beginning of each month. Smith subscribed for 40 shares just 1 month before the end of a 6 months' period. He paid for these shares at the rate of $40 a month. What was the average book value of these 40 shares that was used in the distribution of profits for the 6 months' period? How much was Smith entitled to receive as dividends if the association showed a net gain of $2,631.25, and a book value of $85,160.72?

Shares issued in series.

1. *B* has 20 shares of $100 each in each of 2 series of the Capital City Building and Loan Association. Twenty shares are of a series which is ending its second 6 months' period, and 20 shares are of a series which is ending its first 6 months' period. *B* has paid his dues of $20 a month on each group of shares. The association's rate of profit for the 6 months' period just ended is $5\frac{1}{2}\%$. What are the dividends on each group of shares?

Withdrawal values.

1. Lee paid $20 a month, for 54 months, on 20 shares of stock in the Midway Building and Loan Association. When the 55th payment was due, he withdrew his money for the withdrawal value. The association allowed him the sum of his payments, and simple interest at 5% a year. The value of the stock had been accumulating at 6%, interest convertible monthly. What was the book value of the stock? What was the withdrawal value? How much profit was retained by the association?

2. If, in Problem 1, the interest paid on withdrawals had been calculated at 4%, what would have been the difference between the book value and the withdrawal value?

3. If, in Problem 1, the stock had been accumulating at 7%, interest convertible monthly, and simple interest at 5% was paid on withdrawals, what would have been the difference between the book value and the withdrawal value?

4. A stockholder who has been making payments at the rate of $10 a month for 85 months, withdraws at the date of the 86th payment. The stock has been accumulating at the rate of $5\frac{1}{2}\%$. The association allows 4% simple interest on the payments withdrawn. What is the difference between the book value and the withdrawal value if the stock has a maturity value of $200 a share?

The interest rate from the borrower's standpoint.

1. A certain building and loan association operating on a 7% nominal interest basis will finance the building of your house. If you make payments of $1 a month on each $100 share, your loan will mature with the 79th monthly payment. On the other hand, however, the Mutual Life Insurance Company, through its financial agents in this city, is offering loans on real estate at $5\frac{1}{2}\%$. Suppose that you pay the $5\frac{1}{2}\%$ interest monthly in advance, and invest the difference between this amount and the $1 per share payable to the building and loan association in a sinking fund at 4% interest, payable monthly, will this prove to be a better proposition at the end of 78 months?

2. If, in Problem 1, the difference in the monthly payments were placed in a savings bank at 4%, interest convertible semiannually, would the insurance company's proposition be the better one?

The interest rate from the borrower's standpoint when the interest and dues are considered together as a single sum for the payment of interest and principal.

1. Jones borrowed $1,000 from an association operating on the basis of a 6% nominal interest rate, convertible monthly. Each month he paid $5 dues and $5 interest. His stock matured at the end of 11 years and 3 months, after Jones had made the monthly payment of dues and interest required at that time. What effective rate of interest has Jones paid on his loan? (NOTE.—It will be necessary to use an estimated rate, and to solve by interpolation.)

2. An association is operating on a 7% basis. Smith borrows $1,200, paying $7 interest and $12 dues each month. The 79th payment is $7 interest and $2.40 dues. This is the final payment, and matures the stock. What is the effective interest rate?

Review.

1.* At the end of its fourth year, the Thrifty Building and Loan Association has 500 shares in the first series, 400 in the second, 1,000 in the third, and 800 in the fourth, and its total net earnings for all years amount to $8,364.

(*a*) Compute the rate of earnings under the simple interest partnership plan.

(*b*) Prepare a share statement which includes the profit per series, the book value per series, the book value per share, and such other details as may be necessary.

(*c*) Compute the book value of *M*'s 50 shares of stock in the first series.

(*d*) Compute the withdrawal value of *N*'s 10 shares in the second series, assuming that 4% interest is allowed on withdrawals.

2. The Ypsilanti Building and Loan Association desires that you present to them a statement that they may use to inform their customers as to the comparative cost and returns of their loans compared with loans of a similar nature obtained from other sources. Bank loans in Ypsilanti, on good, salable property, may be obtained by the payment of 7% semiannual interest, and an equal semiannual reduction of the principal. The building and loan association will lend on good, salable property, on condition that the borrower will pay, for each $1,000 borrowed, $5 each month as a repayment of the loan, and $6 interest each month. These payments are to continue until the $5 and the cumulative interest shall be a sum sufficient to repay the loan. Interest is allowed on the repayment of the loan at 8%, compounded semiannually, January and July. Eight per cent simple interest is allowed on each monthly payment until the date of compounding.

3. Distribute a profit of $2,940 on the following series on the partnership plan.

Series	*Paid in Per Share*	*Total Shares*	*Paid in Per Series*	*Average No. Months*	*Total for One Month*	*Profit Per Series*
1	$60	100	....	30 5	....	....
2	48	200	....	24 5	....	....
3	36	300	....	18.5	....	....
4	24	250	....	12.5	....	....
5	12	400	....	6.5	....	....
						$2,940.00

* C. P A., Pennsylvania.

4. How many years will be required to mature stock with a par value of \$100 a share if monthly payments of \$1.55 are made regularly, interest at 6%?

5. If stock with a par value of \$100 a share matures in 7 years, payments being \$1 a share monthly, what effective rate of interest is earned on the investment?

6. A building and loan association organized on the serial plan issued 500 shares of Series A stock, par value \$100 a share, on the first day of its fiscal year, and 500 shares quarterly thereafter for a period of two years. The first of each month, payments of 50 cents a share were made on this stock. At the end of two years it was found that profits for the last quarter-year available for distribution to shareholders amounted to \$729. Find the profit per share for the quarter on the basis of the equated capitals of the respective series.

7. If in Problem 6 dividends have been credited at the rate of 30 cents a share each quarter, find the profits per share if the \$729 were to be distributed on the basis of earning capital.

CHAPTER 36

Permutations and Combinations

Permutation. A permutation is each arrangement which can be made by using all or part of a number of things. The "number of permutations of n things taken r at a time," represented by the symbol ${}_nP_r$, is the number of arrangements of r things that can be formed from n things. Thus, using the three letters a, b, and c taken two at a time, the permutations are ab, ac, ba, bc, ca, and cb. Using all of them at the same time, the permutations are abc, acb, bac, bca, cao, and cba.

Since ${}_nP_r$ is used to denote the number of permutations of n things taken r at a time, its value is determined as follows:

For first place: any one of n things may be chosen;
For second place: any one of the remaining, or $n - 1$, things may be chosen;
For third place: any one of the remaining, or $n - 2$, things may be chosen;
For fourth place: any one of the remaining, or $n - 3$, things may be chosen;
and so forth;
For the last or
rth place: there remains a choice of $n - (r - 1)$ or $n - r + 1$ things.
Therefore, ${}_nP_r = n(n - 1)(n - 2)(n - 3) \cdots (n - r + 1)$.

Take the three letters a, b, and c two at a time. For the first place, there is a choice of 3 letters; for the second place, there is a choice of $3 - 1$ letters; therefore, ${}_3P_2 = 3(3 - 1) = 6$, the number of permutations of three letters taken two at a time, as shown in the first paragraph.

Using the three letters a, b, and c all at the same time, $r = n$. Therefore, the symbol may be expressed ${}_nP_n$ when all of the n things are taken at once, and

$${}_nP_n = n(n - 1)(n - 2)(n - 3) \cdots \text{(until } n \text{ factors are used).}$$

The symbol $n!$, called "factorial n," denotes the product of all integers from n to 1 inclusive, and the expression is abbreviated to ${}_nP_n = n!$. Solving the foregoing example, we have ${}_3P_3 = 3 \cdot 2 \cdot 1 = 6$, the number of permutations of the three letters taken three at a time, as shown in the first paragraph.

NOTE: To avoid confusing the multiplication sign ($\times$) and the sign for quantity (x), use is made of the $\cdot$ placed above the line of writing, and is read "times" or "multiplied by." Also, the

. . . placed on the line of writing indicates omission of the "in-between" factors.

Example 1

Determine the number of three-letter code words that can be made from the letters of the word *bunch*, not repeating a letter in any word.

Solution

The answer is the number of arrangements that can be made from five objects (the letters of the word *bunch*) taken three at a time.

Formula

$$_nP_r = n(n-1) \cdots (n-r+1)$$

Arithmetical Substitution

$$_5P_3 = 5(5-1)(5-2) = 5 \cdot 4 \cdot 3 = 60$$

Example 2

Determine the number of five-letter code words obtainable in the foregoing example.

Solution

The answer is the number of arrangements that can be made from five objects taken all at the same time, or $_nP_n = n!$; and, since $n!$ is the product of all the integers from n to 1, we have

$$n! = n(n-1)(n-2)(n-3)(n-4), \text{ or five factors in all.}$$

Arithmetical Substitution

$$_5P_5 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

Notice that the last factor $(n - r + 1)$ is one more than the difference between the number of things, n, and the number of places, r. Thus, if n is 7, and r is 5, the last factor is 3; also, the number of factors will be equal to r. So we may write the formula $_nP_r = n(n-1)(n-2) \cdots$ (until r factors are used).

Example 3

Five persons enter a doctor's waiting room in which there are seven vacant chairs. In how many ways can they take their places?

Solution

Formula

$$_nP_r = n(n-1)(n-2) \cdots \text{ (until } r \text{ factors are used)}$$

Arithmetical Substitution

$$\begin{aligned} _7P_5 &= 7(7-1)(7-2)(7-3)(7-4) \text{ (five factors)} \\ &= 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2{,}520 \end{aligned}$$

The formula as used above may be expressed as $_nP_r = \dfrac{n!}{(n-r)!}$.

Using the data in Example 3, we have:

$$
\begin{aligned}
{}_7P_5 &= \frac{7!}{(7-5)!} \\
&= \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = 2{,}520
\end{aligned}
$$

Notice that the last two factors cancel the two below the line, and that the remaining factors are the same as in the preceding solution.

Using the three letters a, b, and c, two at a time, we have·

$$
{}_nP_r = \frac{n!}{(n-r)!} = \frac{3!}{(3-2)!} = \frac{3 \cdot 2 \cdot 1}{1} = 6
$$

In the foregoing illustrations, the objects were distinct, that is, there were no repetitions of objects. If the objects are not distinct, the formula is altered to read:

$$
P_n = \frac{n!}{a!b!c!},
$$

where, of the n things, there are a alike, b alike, and c alike.

Example 1

How many permutations may be made of the letters of the word *Illinois?*

Solution

There are eight letters in the word *Illinois*, but three are *i*'s and two are *l*'s. Then we have:

$$
\frac{8!}{3! \cdot 2!} = \frac{40{,}320}{12} = 3{,}360
$$

Example 2

How many permutations may be made of the letters of the word *Indianola?*

Solution

There are nine letters in the word *Indianola*, but two are *a*'s, two are *i*'s, and two are *n*'s. Then we have:

$$
\frac{9!}{2! \cdot 2! \cdot 2!} = \frac{362{,}880}{8} = 45{,}360
$$

Number of ways of doing two or more things together. If a certain thing can be done in m ways, and a second thing can be done in n ways, the two things can be done in succession in $m \cdot n$ ways, or mn ways. The principle can be extended to find the number of ways of doing three or more things together, as $m \cdot n \cdot p \cdot \ldots$ ways.

Example

In how many ways can two positions, the one that of bookkeeper and the other that of stenographer, be filled when there are five applicants for the position of bookkeeper and three applicants for that of stenographer?

Solution

Assuming that all applicants are qualified, there are five ways of filling the position of bookkeeper, and for each of these there is a choice of three stenographers; hence, the two positions can be filled in $5 \cdot 3 = 15$ ways.

For each of the m ways of filling the position of bookkeeper there are n ways of filling the position of stenographer; that is, there are n ways of filling both positions for each way of filling the position of bookkeeper. Therefore, there are in all mn ways of filling the two positions together.

Problems

1. If three dice are thrown together, in how many ways can they fall?

2. There are eight vacant seats to be filled by five persons. In how many ways can they take their places?

3. How many five-place numbers can be made from the digits 1, 2, 3, 4, 5 6, and 7?

4. What is the number of permutations of the letters (*a*) of the word *Indiana;* (*b*) of the word *Illiopolis?*

5. Using three letters at a time, how many permutations can be formed with the letters *abcd?*

6. (*a*) How many permutations of the letters *abcde* can be formed four at a time? (*b*) Five at a time? (*c*) Three at a time?

7. How many permutations may be made of six objects taken: (*a*) six at a time? (*b*) five at a time? (*c*) two at a time?

8. Given the numbers 2, 3, 4, 5, and 6. How many four-place numbers can be formed therefrom?

9. The Greek alphabet contains 24 letters. If no repetition of letters are allowed, how many three-letter fraternities can be named therefrom?

10. A signal man has five flags, no two of which are alike. (*a*) How many different signals can be made by placing them in a row using all five of them each time? (*b*) How many by using three at a time?

Combinations. A combination is a set or selection of r things out of a total of n things without reference to the order within the selection; therefore, *ab* and *ba* are the same combination. The "number of combinations that can be made from a total of n things taken r at a time" is denoted by the symbol ${}_nC_r$. Since each combination can be arranged in more than one way, the number of permutations is denoted by $r!$, and the total number of permutations for all combinations is $r!{}_nC_r$. The total number of permutations of n things taken r at a time is denoted by the symbol ${}_nP_r$, as in previous paragraphs.

Therefore, $$r!\,{}_nC_r = {}_nP_r$$

and $${}_nP_r = \frac{n!}{(n-r)!}$$

Substituting and dividing by $r!$,

$${}_nC_r = \frac{\dfrac{n!}{(n-r)!}}{r!} = \frac{n!}{r!(n-r)!}$$

Example

Find the number of combinations which can be made with the four letters a, b, c, and d taken three at a time.

Solution

$${}_4C_3 = \frac{4!}{3!(4-3)!} = \frac{4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1} = 4$$

These combinations are: *abc*, *abd*, *acd*, and *bcd*. Notice in permutations that the three letters a, b, and c form six permutations, *abc*, *acb*, *bac*, *bca*, *cab*, and *cba*, but there is only one combination *abc*, since all others are merely a rearrangement of the same letters. The addition of the fourth letter makes possible three more combinations.

Example

How many lines can be drawn connecting seven points, no three of which are in the same straight line?

Solution

If we let the points be represented by the letters a, b, c, d, e, f, and g, and any line connecting two of them by the symbol *ab*, *ac*, and so on, we find that *ab* and *ba* is the same line, that *ac* and *ca* is another line, and so forth; therefore, the problem is that of finding the number of combinations of seven objects taken two at a time.

$${}_7C_2 = \frac{7!}{2!(7-2)!} = \frac{7\cdot 6}{2!} = 21$$

NOTE: The factors from *5* to *1* above the line cancel the same factors below the line, leaving the factors indicated in the solution.

If r is large and the difference between n and r is small, the following formula will save considerable work: ${}_nC_r = {}_nC_{n-r}$.

Example

Find the value of ${}_{25}C_{23}$.

Solution

$${}_{25}C_{23} = {}_{25}C_{25-23} = {}_{25}C_2 = \frac{25\cdot 24}{1\cdot 2} = 300$$

Problems

1. How many combinations can be made with the five letters a, b, c, d, and e taken three at a time?

2. If twelve members of an association are available for committee assignments, how many different committees of four each can be selected?

3. You have nine friends that you wish to invite to dinner parties of four guests each. How many dinner parties can you have without having the same company of four twice?

4. A committee consisting of two men and one woman is to be formed from a party of five men and four women. In how many ways can the committee be chosen?

CHAPTER 37

Probability

Probability. One of the principal applications of permutations and combinations is found in the theory of probability. The probabilities for the occurrence of one or more events, in cases in which it is possible to count the number of equally likely ways in which the event can happen or fail, are known as *priori* probabilities.

Counting of some sort is the background of probability. For example, if a coin is tossed, the chances are even between heads and tails. If a die is thrown, the chances of throwing any one of the numbers 1 to 6 is 1 in 6, as there are six surfaces numbered from 1 to 6. The chance of drawing an ace from a well-shuffled pack of 52 cards is evidently 4 in 52.

If an event can occur in m ways and fail in n ways, and if each of these ways is equally likely, then the probability of its occurring is

$$p = \frac{m}{m + n},$$

and the probability of failure in an event is

$$q = \frac{n}{m + n}.$$

Example

Compute the probability of throwing a 3 in the first throw of a die.

Solution

$$p = \frac{m}{m + n} = \frac{1}{1 + 5} = \frac{1}{6}$$

Example

Compute the probability of failing to throw a 3 in the first throw of a die.

Solution

$$q = \frac{n}{m + n} = \frac{5}{1 + 5} = \frac{5}{6}$$

The mathematician computes the ratio of successes to the total number of ways in which the event can occur. For example,

he would say that the chances of throwing a 4 in one toss of a die is "one chance in six." The average person usually calculates the ratio of the successes to the failures, and would say "the odds are five to one against throwing a 4."

Problems

1. A bag contains ten black balls and fifteen white ones. What is the probability that a ball drawn at random will be black?

2. A box contains six times as many black balls as white ones and one ball is drawn at random. What is the probability that the ball drawn will be black?

3. If you are to win a prize valued at $12.00 by throwing an ace in a single throw with a die, what is the value of your expectation?

Permutations and combinations in probability. The following examples will illustrate the statement made at the beginning of this chapter.

Example

What is the probability of obtaining a 6 if two dice are tossed?

Solution

Under permutations we learned that a succession of acts can be performed together in as many ways as the result of their continued product. Since each die has 6 faces, the two dice can fall in 6×6, or 36, ways. In these 36 ways, the sum 6 can appear in any one of the following 5 ways: 5 and 1, 1 and 5, 4 and 2, 2 and 4, 3 and 3. Therefore, the probability of throwing a 6 is $\frac{5}{36}$.

Example

If two cards are drawn from a complete deck of 52 cards, what is the probability that both are hearts?

Solution

First is the determination of the number of combinations of 52 objects taken 2 at a time.

$$_nC_r = \frac{n!}{r!(n-r)!} = \frac{52!}{2!(52-2)!} = 1,326$$

Second is the determination of the number of selections of two hearts.

$$_nC_r = \frac{13!}{2!(13-2)!} = 78$$

Therefore, the probability of selecting two hearts is $\frac{78}{1326}$ or $\frac{1}{17}$.

Example

Two prizes are offered in a lottery of 20 tickets. What is your probability of winning a prize if you hold five tickets?

Solution

First, determine the number of ways in which five tickets can be selected.

$$_nC_r = \frac{20!}{5!(20-5)!} = 15{,}504$$

If you hold the two prize tickets, the remaining tickets may be any three of the remaining 18, so the number of selections containing both prizes is

$$_nC_r = \frac{18!}{3!(18-3)!} = 816$$

Next, determine the number of selections containing the first prize and not the second.

$$_nC_r = \frac{18!}{4!(18-4)!} = 3{,}060$$

The number of selections containing the second prize and not the first is evidently the same, 3,060.

Therefore, the probability of winning a prize is

$$\frac{816 + (2 \times 3{,}060)}{15{,}504} = \frac{6{,}936}{15{,}504} = \frac{17}{38}$$

Example

In the foregoing example, what is the probability that you will not win a prize?

Solution

As two of the tickets are winners, 18 are not winners, and this is the number from which five tickets must be selected, then,

$$_nC_r = \frac{18!}{5!(18-5)!} = 8{,}568$$

As before,

$$_nC_r = \frac{20!}{5!(20-5)!} = 15{,}504$$

The probability that there will be no winner is

$$\frac{8{,}568}{15{,}504} = \frac{21}{38}$$

Checking the answer to the preceding problem, we have:

$$1 - \tfrac{21}{38} = \tfrac{17}{38}$$

Problems

1. A complete deck of cards numbers 52 and is made up of 13 cards in each of the four suits. If four cards are drawn, find the following probabilities:

(*a*) That all are hearts;
(*b*) That there is one card of each suit;
(*c*) That there are two diamonds and two clubs.

2. (*a*) In a single throw of two dice, what is the probability of throwing a ten? (*b*) What would be the probability in a single throw of three dice?

3. If you toss six coins, what is the probability that there are four heads and two tails?

4. The cash drawer contains five ten-dollar bills, six five-dollar bills, and seven one-dollar bills. How many different sums may be formed with three bills taken out at random?

Compound events. The joint occurrence of two or more simpler events in connection with one another is called a *compound event.* If two or more events occur without influencing one another, they are said to be independent; but if any one of them does affect the occurrence of the others, they are said to be dependent. When the occurrence of any one of the events excludes the occurrence of any other on that occasion, the events are said to be mutually exclusive.

Independent events. The probability that n independent events will happen favorably on a given occasion (when all of them are in question) is the product of their separate probabilities. If the separate probabilities that an event can occur favorably are represented by $p_1, p_2, \ldots p_n$, and P equals the probability that all these events will happen together at a given trial, then,

$$P = p_1 \times p_2 \times \cdots p_n$$

Example

What is the probability that 2, 3, and 4 are thrown in succession with a die?

Solution

The probability of getting 2 is $\frac{1}{6}$, that of getting 3 is $\frac{1}{6}$, and that of getting 4 is $\frac{1}{6}$. As these are the probabilities of independent events, the joint probability will be $\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216}$.

Example

What is the probability of getting 2, 3, and 4 in one throw with three dice?

Solution

Consider the three dice as being thrown separately. Then there are 3 chances in 6 of getting the first number, 2 chances in 6 of getting the second number, and one chance in 6 of getting the third number. Since these events are all independent of one another, the joint probability will be $P = \frac{3}{6} \times \frac{2}{6} \times \frac{1}{6} = \frac{1}{36}$.

Example

From a box containing 5 brown marbles and 4 green marbles, 3 marbles are drawn. What is the probability that all three will be brown?

Solution

Consider each drawing an independent event. Since there are at first 9 marbles and 5 are brown, the probability on the first drawing will be $\frac{5}{9}$. If a brown marble has been drawn, then on the second drawing the probability will be $\frac{4}{8}$. Now, if 2 brown marbles have been drawn, on the third drawing the probability will be $\frac{3}{7}$. Since these three probabilities have been independent events, the joint probability is

$$P = \frac{5}{9} \times \frac{4}{8} \times \frac{3}{7} = \frac{5}{42}$$

Mutually exclusive events. Let the number of mutually exclusive events be represented by p_1, p_2, . . . p_n. The probability that some one of these events will occur is equal to their sum; therefore, $P = p_1 + p_2 + \cdots p_n$.

Example

Three horses are entered in a race. Snowball's chances of winning are $\frac{1}{6}$, Thunderbolt's chances are $\frac{1}{2}$, and Fleetwind's is $\frac{1}{4}$. What is the probability that the race will be a tie?

Solution

The winning of the race by Snowball, Thunderbolt, or Fleetwind forms a set of mutually exclusive events, since only one can be the winner. The probability that one of them wins the race is

$$P = \frac{1}{6} + \frac{1}{2} + \frac{1}{4} = \frac{11}{12}$$

Since the race may be a tie, that probability is

$$q = 1 - \frac{11}{12} = \frac{1}{12}$$

Example

If my chance of completing a certain engagement is $\frac{1}{4}$, and your chance of completing it is $\frac{2}{3}$, what is the probability that the engagement will be completed if we both work independently of one another?

Solution

If we work together to complete the engagement,

$$p_1 = \frac{1}{4} \times \frac{2}{3} = \frac{2}{12}$$

If I complete the engagement and you fail to complete it,

$$p_2 = \frac{1}{4} \times \left(1 - \frac{2}{3}\right) = \frac{1}{12}$$

If you complete it and I fail,

$$p_3 = \frac{2}{3} \times \left(1 - \frac{1}{4}\right) = \frac{6}{12}$$

The sum of $p_1 + p_2 + p_3 = \frac{2}{12} + \frac{1}{12} + \frac{6}{12} = \frac{3}{4}$, the probability that the engagement will be completed by one or the other of us.

This result can be checked by assuming that both will fail:

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{2}{3}\right) = \frac{1}{4}$$
$$\frac{4}{4} - \frac{1}{4} = \frac{3}{4}$$

Example

From a box containing 5 brown marbles and 4 green marbles, 2 marbles are drawn at random. What is the probability that both are of the same color?

Solution

The probability that the 2 are brown is determined as shown on the next page.

$$\frac{5!}{2!(5-2)!} = 10 \quad \text{and} \quad \frac{9!}{2!(9-2)!} = 36; \quad \frac{10}{36} = \frac{5}{18}$$

The probability that the 2 are green is determined in the same manner.

$$\frac{4!}{2!(4-2)!} = 6 \quad \text{and} \quad \frac{9!}{2!(9-2)!} = 36; \quad \frac{6}{36} = \frac{3}{18}$$

These two events are mutually exclusive; hence, the probability that both marbles are of the same color is

$$\tfrac{5}{18} + \tfrac{3}{18} = \tfrac{4}{9}$$

The probability that there will be one of each color is

$$\frac{5 \times 4}{\dfrac{9!}{2!(9-2)!}} = \frac{5 \times 4}{36} = \frac{5}{9}$$

Problems

1. A and B engage in a game of checkers. The probability that A will win a game is $\frac{2}{3}$ and that B will win a second game is $\frac{1}{3}$. What is the probability that both win?

2. From a bowl containing 5 red marbles and 6 white ones, 4 marbles are drawn at random. What is the probability that they are all white?

3. In Problem 2, what is the probability that the 4 marbles drawn at random are red?

4. In Problem 2, determine the probability that of the 4 marbles drawn, 2 are red and 2 are white.

5. If three cards are drawn from an ordinary pack of playing cards, what is the probability that all three will be spades?

6. In Problem 5, what is the probability that all three cards are black?

7. What is the probability that all three will be of the same suit?

8. If the cards are replaced after each drawing, what is the probability that the first three are hearts?

9. At an election, 500 of the registered voters in the precinct cast their ballots. Two hundred voted in favor of a certain amendment and 300 voted against it. If five voters are chosen at random, what is the probability that they all voted for the amendment?

Empirical probability. The practical application of probabilities leads to a consideration of probabilities which are derived from experience. These are called *empirical* probabilities. The connection between probability and statistics, a subject devoted to the analysis and interpretation of data, is found in empirical probability.

Example

The following table gives the average daily sales of 92 market gardeners in a certain public market.

Average Daily Sales	2 50–7 49	7 50–12.49	12 50–17 49	17 50–22.49	22 50–27.49	27 50–32.49	32 50–37 49	37 50–42 49	42 50–47 49	47.50–52 49
Number of Gardeners	2	8	27	21	16	3	11	2	1	1

What is the probability of a person engaging in market gardening in this market of having average daily sales of less than \$17.50?

Solution

The table shows that of 92 gardeners, the number who make less than \$17.50 average daily sales is $2 + 8 + 27 = 37$. The required probability is, therefore, $\frac{37}{92}$, expressed decimally as 0.402.

Problems

1. A study of market gardening showed that 100 gardeners had marketed in a particular market as follows:

Years	*Number of Gardeners*
1 to 5	31
6 to 10	22
11 to 15	10
16 to 20	19
21 to 25	6
26 to 30	3
31 to 35	5
36 to 40	3
41 to 45	1
	100

What is the probability of a gardener being in this market for 20 years or less?

2. A survey among farmers to determine their average income disclosed the following facts:

Income Range	*Per Cent of Farmers*
Negative income (Loss)	4.55
0 to \$ 500	12.53
\$ 500 to \$ 1,000	21.89
\$ 1,000 to \$ 1,500	17.01
\$ 1,500 to \$ 2,000	14.69
\$ 2,000 to \$ 3,000	13 31
\$ 3,000 to \$ 4,000	7.66
\$ 4,000 to \$ 5,000	3.25
\$ 5,000 to \$ 7,500	3.17
\$ 7,500 to \$10,000	0.93
\$10,000 and over	1.01
	100.00

Based on the above survey, what is C's probability of making $2,000 to $3,000 a year if he engages in farming?

3. A manufacturer of electric light bulbs made a test with 200 bulbs of uniform design. The results were tabulated as follows:

Life in Hours	400 to 500	500 to 600	600 to 700	700 to 800	800 to 900	900 to 1,000	1,000 to 1,100	1,100 to 1,200	1,200 to 1,300	1,300 to 1,400	1,400 to 1,500	1,500 to 1,600
Number of Bulbs	2	4	7	16	23	27	37	31	24	18	8	3

What is the probability that a bulb will burn out in less than 800 hours, based on the experience of the foregoing test?

4. If three new bulbs are placed in operation at the same time, what is the probability that all of them will last 1,200 hours?

HINT: Cube of a single probability.

5. Find the probability that a bulb will be "alive" between 900 and 1,300 hours.

CHAPTER 38

Probability and Mortality

Life insurance. Life insurance is based upon probabilities determined by the actual study of large collections of mortality statistics. If an event has happened m times in n possible cases (where n is a large number), then, in the absence of further knowledge, it may be assumed for many practical purposes that $\frac{m}{n}$ is the best estimate of the probability of the event and that confidence in this estimate may increase as n increases. The fraction $\frac{m}{n}$ is called the *frequency ratio*.

According to the American Experience Table of Mortality (see page 536), of 69,804 men living at age 50, the number living ten years later will be 57,917. The probability that a man aged 50 will live ten years is taken to be

$$\frac{57{,}917}{69{,}804} = .8297$$

Mortality table. Application of the theory of probability is made in the study of problems involving the duration of human life, such as life insurance, life annuities, pensions, and so forth. Tables that show the number of deaths expected to occur during a given age are used in the solution of these problems. Census records and vital statistics gathered by governmental agencies are the basis of some mortality tables. Others are based upon the records of life insurance companies. Results based upon mortality tables are applicable only to large groups of individuals.

The table on page 454 is taken from the American Experience Table. (The entire table is given in Table 7, in the Appendix.)

Column (1) is the age column and contains the age of 100,000 people or their survivors.

Column (2) indicates the number of people living at the beginning of the year designated on the same line in column (1). The table starts with 100,000 people alive at age 10 and at age 11 shows that only 99,251 have survived. The number that have died in the interval, 749, is shown in column (3).

Column (4) is $\frac{749}{100{,}000} = .00749$ at age 10, and so on for each age.

Column (5) is $\frac{99{,}251}{100{,}000} = .99251$ at age 10, and so on for each age.

AMERICAN EXPERIENCE TABLE OF MORTALITY

(1)	(2)	(3)	(4)	(5)
Age	Number Living	Number of Deaths	Yearly Probability of Dying	Yearly Probability of Living
x	l_x	d_x	q_x	p_x
10	100,000	749	0.007490	0 992510
11	99,251	746	0 007516	0.992484
12	98,505	743	0.007543	0.992457
13	97,762	740	0.007569	0.992431
14	97,022	737	0.007596	0.992404
15	96,285	735	0 007634	0.992366
16	95,550	732	0 007661	0.992339
17	94,818	729	0.007688	0.992312
18	94,089	727	0 007727	0.992273
19	93,362	725	0.007765	0.992235
20	92,637	723	0 007805	0 992195
...	...	...	...	...
30	85,441	720	0.008427	0 991573
...	...	...	...	...
50	69,804	962	0.013781	0.986219
...	...	...	...	...
70	38,569	2,391	0.061993	0.938007
...	...	...	...	...
90	847	385	0.454545	0.545455
...	...	...	...	...
95	3	3	1.000000	0.000000

Notation. For convenience, the letter l is used to designate entries in the "living" column. Subscripts appended to the l denote specific entries in this column; thus, l_{15} indicates the number living at age 15.

It is customary to refer to the age as x, that is, any age; therefore, l_x would apply to any entry in column (2).

The number dying is denoted by d, which with the subscript, as in d_{15}, indicates the number dying between the age indicated and the next. Since x stands for any year, d_x indicates any entry in column (3).

The probability of a person dying is expressed by q. With the subscript, as in q_{15}, it denotes the probability of a person of the age indicated dying before reaching the following age. Similarly, q_x stands for the probability of a person aged x dying before reaching age $x + 1$. This probability is shown in column (4).

The probability of living is denoted by the letter p, and p_{15} denotes the probability that a person aged 15 will live to become age 16. This probability is shown by the table, column (5), to be .992366. The symbol p_x denotes the probability that a person aged x will live to age $x + 1$.

For convenience, the symbols already explained and others based on them are shown in the following summation:

x = a person or a life aged x years
l_x = the number of persons living at age x
l_{x+1} = the number living at age $x + 1$
l_{x+n} = the number living at age $x + n$
d_x = the number of persons dying in the age interval x to $x + 1$
d_{x+1} = the number dying in the age interval $x + 1$ to $x + 2$
p_x = the probability that a person of age x will live one year
q_x = the probability that a person of age x will die within one year
${}_np_x$ = the probability that a person of age x will live at least n years
$|_nq_x$ = the probability that a person of age x will not live n years
${}_n|q_x$ = the probability that a person of age x will die within one year after reaching the age of $x + n$
p_{xy} = the probability that two persons, of ages x and y, respectively, will live at least a year
${}_np_{xy}$ = the probability that two persons, of ages x and y, respectively, will live at least n years

Probability of living. On page 453 was shown the probability that a man aged 50 will live ten years is taken to be $\frac{57{,}917}{69{,}804} = .8297$.

The probability that a person aged 50 will survive 10 years is equal to the ratio between the number living at age 60 and the number living at age 50. Expressed as a formula,

$$ {}_{10}p_{50} = \frac{l_{60}}{l_{50}} $$

The probability that a person aged x will live to age $x + 1$ is equal to the ratio of the number of people living at age $x + 1$ to the number living at age x, or

$$ p_x = \frac{l_{x+1}}{l_x} $$

Example

What is the probability that a person of age 30 will live at least a year?

Solution

$$p_x = \frac{l_{x+1}}{l_x} = \frac{l_{31}}{l_{30}} = \frac{84721}{85441} = .991573$$

The probability that a person will live longer than one year or n years is expressed by the formula

$$_np_x = \frac{l_{x+n}}{l_x}$$

Example

Determine the probability that a person age 40 will live to be 60.

Solution

$$_np_x = \frac{l_{x+n}}{l_x} = \frac{l_{60}}{l_{40}} = \frac{57917}{78106} = .7415$$

Probability of dying. The probability that a person aged x will die within a year is expressed by

$$q_x = \frac{d_x}{l_x}$$

Example

What is the probability that a person aged 40 will die within one year?

Solution

$$q_{40} = \frac{d_{40}}{l_{40}} = \frac{765}{78106} = .009794$$

The probability that a person aged x will not live to age $x + n$ is ascertained by the following formula:

$$|_nq_x = \frac{l_x - l_{x+n}}{l_x}$$

Example

What is the probability that a person aged 30 will not live to age 40?

Solution

$$|_{10}q_{30} = \frac{l_{30} - l_{30+10}}{l_{30}}$$

$$= \frac{85441 - 78106}{85441}$$

$$= .085849$$

Since the sum of the probability of living and the probability of dying equals 1, or certainty, the foregoing example may be solved as follows:

$$|_n q_x = 1 - {}_n p_x$$

$$|_{10} q_{30} = 1 - {}_{10}p_{30} = 1 - \frac{l_{40}}{l_{30}}$$

$$= 1 - \frac{78106}{85441}$$

$$= .085849$$

The probability that a person aged x will die within one year after reaching age $x + n$ is ascertained by the following formula:

$$_{n|}q_x = \frac{d_{x+n}}{l_x}$$

Example

What is the probability that a person aged 30 will die within one year after reaching age 40?

Solution

$$_{10|}q_{30} = \frac{d_{40}}{l_{30}} = \frac{765}{85441} = .008959$$

Joint life probabilities. The probability that two persons (x) and (y) will survive at least a year is denoted by p_{xy}. As the probabilities of life, or of death, of two or more persons are assumed to be independent of each other, it follows that

$$p_{xy} = \frac{l_{x+1} \cdot l_{y+1}}{l_x \cdot l_y}$$

Example

A husband and wife are aged 37 and 32, respectively. What is the probability that both will be alive at the end of 20 years?

Solution

The probability that the husband will be alive at the end of 20 years is

$\frac{62104}{80353}$, or 0.7729

The probability that the wife will be alive at the end of 20 years is

$\frac{67841}{84000}$, or 0.8076

Since the probability of two separate and distinct events is the product of the probabilities of each event, the probability that both will be alive at the end of 20 years is

$$0.7729 \cdot 0.8076 = 0.6242$$

NOTE: Since (x) is used in life insurance to denote the age of a person, it is confusing to use it to represent *times*, or *multiplied by;* therefore, multiplication is indicated by the period placed above the line. The formula may also be written

$$p_{xy} = \frac{l_{x+1} \cdot l_{y+1}}{l_{xy}}$$

Problems

1. What is the probability that a child aged 12 will die between the ages 15 and 16?

2. A father and son are aged 35 years and 13 years, respectively. Find the probability that both will be living on the son's twenty-first birthday.

3. What is the probability that a man aged 35 will die within 5 years? What is the probability that he will die in the year after he reaches age 40?

4. A husband and wife are aged 42 and 40, respectively. The husband has purchased an annuity, the first payment to be on his 65th birthday. What is the probability that both will be living at that time?

5. What is the probability that neither will be living when the first payment of the annuity described in Problem 4 is due?

6. If the annuity described in Problem 4 is payable for 20 years, what is the probability that both husband and wife will be living when the twentieth payment is due?

7. A husband and wife are aged 26 and 24, respectively, at the date of their marriage. What is the probability that they will live to celebrate their golden wedding anniversary?

8. Y is aged 40 and Z is aged 35. Calculate the following:

(*a*) That Y will survive the first year but Z will not.
(*b*) That both will survive one year.
(*c*) That Z will survive the first year but Y will not.
(*d*) That both will survive 15 years.
(*e*) That both will die during the first year.

9. Find the probability that a person aged 50 will live to age 70.

10. Find the probability that a person aged 25 will not live to age 35. What is the probability that this person will die between the ages of 35 and 36?

CHAPTER 39

Life Annuities

Factors involved. In Chapter 30 it is shown that the present value of a sum of money payable n years in the future depends upon the rate of interest which can be earned.

If the payment of this sum of money at a future time is contingent on some person being alive at such future time, the present value depends upon the rate of interest and also upon the probability that the person will be living. For example, if two equally good insurable risks aged 25 and 65, respectively, are to receive $1,000 each upon attaining ages 35 and 75, respectively, the present value of the promised payment to the person aged 25 would be relatively much greater than to the person aged 65.

Pure endowment. A pure endowment contract promises to pay to the holder thereof a definite sum of money if he is living at the end of a specified period, but nothing to his beneficiaries if he fails to survive this period.

The present value of an n-year pure endowment of 1 to a person now aged x is expressed by the symbol ${}_nE_x$, which is equivalent to the present value of 1 to be received at the end of n years, multiplied by the probability ${}_np_x$ that a person aged x will survive n years.

If ${}_nE_x$ denotes the present value of an n years' pure endowment to a person of age x, we have

$$ {}_nE_x = \left[\frac{1}{(1+i)^n}\right] {}_nP_x \qquad {}_nP_x = \frac{l_{x+n}}{l_x} $$

OR

$$ {}_nE_x = \frac{\left(\frac{1}{(1+i)^n}\right) l_{x+n}}{l_x} $$

Example

A person aged 20 is to receive $5,000 upon attaining age 25. Find the present value of the probability, interest at $3\frac{1}{2}\%$.

Solution

$$ 5{,}000\,{}_5E_{20} = 5{,}000 \left[\frac{\frac{1}{(1.03\frac{1}{2})^5}\, l_{25}}{l_{20}}\right] $$

The probability that the person will receive the money is

$$_5p_{20} = \frac{l_{25}}{l_{20}} = \frac{89032}{92637} = .9610846$$

$$\frac{1}{(1.03\frac{1}{2})^5} = .8419732, \text{ the present value of 1 for 5 years at } 3\tfrac{1}{2}\%$$

The present value to the person aged 20 is, then,

$$\$5{,}000 \times .8419732 \times .9610846 = \$4{,}046.04$$

Problems

1. A girl aged 10 is to receive \$5,000 upon attaining age 18. Find the present value of the inheritance, interest at $3\frac{1}{2}\%$.

2. A persor, aged 20, is to receive \$10,000 upon reaching age 30. Find the present value of his expectation on the basis of $3\frac{1}{2}\%$ interest and the American Experience Table of Mortality.

3. Find the present value of a pure endowment of \$2,000 to a person aged 30 payable if he reaches the age of 60, on a $3\frac{1}{2}\%$ basis.

Life annuity. A series of periodical payments during the continuance of one or more lives constitutes a life annuity. The simplest form of a life annuity to a person age x is the payment of 1 at the end of each year so long as the person now aged x lives. Such an annuity consists of the sum of pure endowments of 1 each year. The symbol for a life annuity is a_x; therefore, $a_x = {}_1E_x + {}_2E_x + {}_3E_x \cdots + {}_nE_x \cdots$ to table limit.

Substituting these values gives:

$$a_x = \frac{vl_{x+1} + v^2l_{x+2} + v^3l_{x+3} \cdots + v^nl_{x+n}}{l_x} \cdots \text{ to table limit.}$$

Example

Find the value of a life annuity of \$1,000 a year to a person now aged 90, interest at $3\frac{1}{2}\%$.

$$a_{90} = \frac{((1.035)^{-1}l_{91}) + ((1.035)^{-2}l_{92}) + ((1.035)^{-3}l_{93}) + ((1.035)^{-4}l_{94}) + ((1.035)^{-5}l_{95})}{l_{90}}$$

Present values are found in Table 3. Values of l_{90}, l_{91}, and so forth, are found in Table 7.

Substituting all the indicated values and solving, we have

$$a_{90} = .8738$$
$$\$1{,}000 \times .8738 = \$873.80$$

Problem

A life pension of \$500 a year, payable at the end of each year, is granted to a person now aged 91. What is the present value of this pension, interest at $3\frac{1}{2}\%$?

Commutation columns. In the examples and in the preceding problem, the computations are not particulary arduous, because the

age of the annuitant made it necessary to make only a few computations. But, in cases where the annuitant is younger—for example, age 20—it is evident that a great amount of work would be required in order to solve the problem. Much of this computation may be eliminated by the use of tables called "commutation columns" (see Table 8).

The first column of this table, the D_x column, has been constructed of the products of similar v's and l's and the product denoted by the letter D. The computation was therefore reduced to the addition of the values found in the D_x column. To save time in adding these values, another commutation column was formed, containing the sums of all the D's from any particular value of D_x to the table limit. This is the N_x column of Table 8.

Therefore, the work is materially reduced by using the tables and the formula:

$$a_x = \frac{N_{x+1}}{D_x}$$

The solution to the example on page 460 now becomes:

$$a_x = \frac{N_{x+1}}{D_x} = \frac{N_{90+1}}{D_{90}} = \frac{33.47}{38.3047} = .8738$$

$$\$1{,}000 \times .8738 = \$873.80$$

Using the commutation tables, the present value of the pure endowment on page 459 may be found by the formula

$$_nE_x = \frac{D_{x+n}}{D_x}$$

Substituting the values for the example on page 459, we have

$$_5E_{20} = \frac{37673.6}{46556.2} = .809207$$

and

$$\$5{,}000 \times .809207 = \$4{,}046.04$$

From the foregoing formula it may be found that 1 at age x will purchase an n-year pure endowment of

$$\frac{D_x}{D_{x+n}}$$

Problems

1. What is the present value of a life annuity of \$3,000 to a person aged 30, interest at $3\frac{1}{2}\%$?

2. Find the value of a life annuity of \$2,500 at $3\frac{1}{2}\%$ to a person aged 35.

Life annuities due. The principles of annuities apply in life insurance. The preceding illustration was that of a life annuity where the payment was made at the end of each year. When the payments are to be made at the beginning of each year, the life annuity is a *life annuity due.* Actuaries use the symbol $\mathfrak{a}_x$ to represent the present value of an annuity; and, since an annuity due differs from an ordinary annuity by an additional payment made at the beginning of the period, the present value of an annuity due is $1 + a_x$, and the symbol becomes

$$\mathfrak{a}_x = 1 + a_x$$

using a different type "a" from that used in the ordinary life annuity. Since the different type "a" is somewhat difficult to make, the regular a may be used and distinguished by a bar over it, thus,

$$\bar{a}_n = 1 + a_x$$

Use of commutation table. Since $a_x = \dfrac{N_{x+1}}{D_x}$, the life annuity due formula may be written as

$$\bar{a}_x = 1 + \frac{N_{x+1}}{D_x}$$

and if for 1 we substitute $\dfrac{D_x}{D_x}$, we have

$$\bar{a}_x = \frac{D_x}{D_x} + \frac{N_{x+1}}{D_x}, \qquad \text{or} \qquad \bar{a}_x = \frac{D_x + N_{x+1}}{D_x}$$

which is equivalent to

$$\bar{a}_x = \frac{N_x}{D_x},$$

the formula for the present value of a life annuity due of 1 payable to a person aged x. The values may be obtained from the commutation table.

Example

Find $\bar{a}_{30}$

$$\bar{a}_x = \frac{N_x}{D_x}$$

$$\bar{a}_{30} = \frac{N_{30}}{D_{30}}$$

$$\bar{a}_{30} = \frac{596804}{30440.8} \text{ from the table}$$

$\bar{a}_{30} =$ 19.6054, also shown in the table in the $1 - a_x$ column.

Problems

(Use the commutation table.)

1. Find N_{20}.
2. Find D_{35}.
3. If $x = 75$, find D_x.
4. Find N_x if $x = 22$.
5. When $D_x = 25630.1$, what is the value of x?
6. When $N_x = 208510$, what is the value of x?
7. At what age does $N_x = 157{,}255$?
8. At what age does $D_x = 1987.87$?
9. What is the difference in value between $\ddot{a}_{40}$ and a_{40}?
10. Find (1) $\ddot{a}_{30}$; (2) a_{30}.

Deferred annuity. When the first payment under a life annuity is to be made after the lapse of a specified number of years (contingent upon the annuitant (x) being alive), instead of being made a year after the payment of the single premium, the annuity is *deferred.*

Since under an ordinary annuity the first payment is made at the end of one year, then if an annuity is *deferred* n *years*, the first payment is made at the end of $n + 1$ years; but an annuity providing for *the first payment at the end of* n *years* is deferred $n - 1$ years, for the annuity is entered upon at the end of $n - 1$ years, and the first payment is not made until one year later; and it is a deferred life annuity due.

The present value of a life annuity of \$1.00 deferred for n years is expressed by the symbol

$$_{n|}a_x$$

but in n years the annuitant's age will be $x + n$, and the value of the annuity will be a_{x+n}; and, since it is desired to find the value of this annuity *now*, we discount it by multiplying a_{x+n} by the regular present-value symbol, v^n.

However, three factors are to be considered as follows:

(*a*) The value of the life annuity, a_{x+n};
(*b*) The present value of 1 in n years, v^n;
(*c*) The probability that the person aged x will be living n years from now, $_np_x$.

Therefore, the formula becomes:

$$_{n|}a_x = (a_{x+n})(v^n)(_np_x)$$

and, making substitutions so that the solution may be obtained from the commutation table, we have

$$_{n|}a_x = \frac{N_{x+n+1}}{D_x}$$

Example

What single premium will a person aged 30 have to pay to obtain a life annuity of \$2,500, so that he will receive his first annuity payment at the end of his 46th year?

Solution

$${}_{n|}a_x = \frac{N_{x+n+1}}{D_x}$$

$${}_{15|}a_{30} = \frac{N_{30+15+1}}{D_{30}}$$

$${}_{15|}a_{30} = \frac{N_{46}}{D_{30}}$$

From the commutation tables, it is found:

$${}_{15|}a_{30} = \frac{253745}{15773.6}$$

$$= 16.08669$$

$$\$2{,}500 \times 16.08669 = \$40{,}216.73$$

Deferred life annuity due. The first payment of an annuity due would be made 1 year before that of an ordinary deferred annuity; therefore, the deferred life annuity due is the equivalent of an ordinary life annuity deferred for $n - 1$ years, and the formula is

$${}_n\ddot{a}_x = {}_{n-1|}a_x \qquad \text{or} \qquad {}_n\ddot{a}_x = \frac{N_{x+n}}{D_x}$$

Example

Y is aged 55, and he desires to purchase a life annuity of \$2,500, the first payment to be made at age 65. What is the single premium payment?

Solution

$${}_n\ddot{a}_x = \frac{N_{x+n}}{D_x}$$

and

$${}_{10}\ddot{a}_{55} = \frac{N_{55+10}}{D_{55}}$$

$$= \frac{48616.4}{9733.40}$$

$$= 4.9948$$

$$\$2{,}500 \times 4.9948 = \$12{,}487.00$$

Problems

1. A child 15 years of age is to receive \$2,400 a year for life, the first payment to be made at age 21. Calculate the value of this annuity at $3\frac{1}{2}\%$.

2. What single premium payment will a person aged 35 have to pay to obtain a life annuity of \$3,000 from which he will receive his first annuity payment at age 60?

Temporary life annuities. A temporary life annuity continues for n years, contingent on the annuitant living that long; hence, it is not an annuity certain. The symbol for a temporary life annuity is $a_{x\overline{n}|}$, and the formula is

$$a_{x\overline{n}|} = \frac{N_{x+1} - N_{x+n+1}}{D_x}$$

Example

Find the present value of a life annuity of $2,000 for 25 years, to a person aged 40.

Solution

$$a_{x\overline{n}|} = \frac{N_{x+1} - N_{x+n+1}}{D_x}$$

$$a_{x\overline{n}|} = \frac{N_{41} - N_{66}}{D_{40}}$$

$$= \frac{324440 - 43343.1}{19727.4}$$

$$= 14.24906$$

$$\$2{,}000 \times 14.24906 = \$28{,}498.12$$

Temporary annuities due. The present worth of a temporary life annuity due, also termed an *immediate temporary annuity*, is equivalent to the difference between the present worth of a whole life annuity due and a deferred life annuity due, and may be expressed as

$$\bar{a}_{x\overline{n}|} = \bar{a}_x - {}_n|\bar{a}_x$$

Substituting values, the formula for use with commutation tables becomes

$$\bar{a}_{x\overline{n}|} = \frac{N_x - N_{x+n}}{D_x}$$

Example

Y buys a temporary life annuity of $1,200 for his widowed mother aged 50. Payments are to begin at once and to continue until age 75. What is the present value of this annuity due?

Solution

$$\bar{a}_{x\overline{n}|} = \frac{N_x - N_{x+n}}{D_x}$$

$$\bar{a}_{50\ 25} = \frac{N_{50} - N_{50+25}}{D_{50}}$$

$$= \frac{181663 - 11728.9}{12498.6}$$

$$= 13.59625$$

$$\$1{,}200 \times 13.59625 = \$16{,}315.50$$

Problems

1. Find the present value of a temporary life annuity of $1,500 for 5 years to a person aged 65.

2. Find the values of $a_{20\,\overline{10}|}$, $a_{15\,\overline{20}|}$, and $a_{35\,\overline{10}|}$.

3. Find the values of ${}_{10|}a_{20}$, ${}_{20|}a_{15}$, and ${}_{10|}a_{35}$.

Life annuities with payments *m* times a year. Annuity contracts often provide that payments shall be made more frequently than once a year, such as quarterly or monthly, the latter being more common. For an annuity payable m times a year, the symbol $a_x^{(m)}$ is used to denote its present value, and the formula used to determine the value when the payments are made at the end of the period is

$$a_x^{(m)} = a_x + \frac{m-1}{2m}$$

Example

Find the present value of a life annuity of $600 a year payable monthly, the first installment to be paid in one month, for a person 35 years of age.

Solution

$$a_x^{(m)} = a_x + \frac{m-1}{2m}$$

Substituting values,

$$a_{35}^{(12)} = a_{35} + \tfrac{11}{24}$$

$$a_{35} = \frac{N_{36}}{D_{35}} = \frac{432326}{24{,}544.7}$$

$$= 17.6143$$

$$\tfrac{11}{24} = .4583$$

$$17.6143 + .4583 = 18.0726$$

$$\$600 \times 18.0726 = \$10{,}843.56$$

If the payments are made at the beginning of the period, they constitute an annuity due, and its present value will be:

$$\ddot{a}_x^{(m)} = \ddot{a}_x - \frac{m-1}{2m}$$

For a deferred life annuity of 1 a year, payable in m installments a year, the present value is

$${}_{n|}a_{xn}^{(m)} = \frac{D_{x+n}}{D_x} \times a_{x+n}^{(m)}$$

For a temporary life annuity for n years, payable in m installments a year, the present value is

$$a_{x\overline{n}|}^{(m)} = a_{x\overline{n}|} + \frac{m-1}{2m}(1 - {}_nE_x)$$

Example

The value of a temporary life annuity of \$180 a year payable annually for 15 years to a person aged 45 is \$1,873.96.

What is the present value of an annuity of the same annual rent if paid in monthly installments of \$15 each, the first payment one month hence?

Solution

$$\$180a_{45\,\overline{15}|} = \$1{,}873.96$$
$$a_{45\,\overline{15}|} = 10.4109$$
$$a_{45\,\overline{15}|}^{(12)} = a_{45\,\overline{15}|} + \tfrac{11}{24}(1 - {}_{15}E_{45})$$
$${}_{15}E_{45} = \frac{D_{60}}{D_{45}} = \frac{7351.65}{15773.6} = .46606$$

Therefore,

$$a_{45\,\overline{15}|}^{(12)} = 10.4109 + \tfrac{11}{24}(.53394)$$
$$= 10.4109 + .2447 = 10.6556$$
$$\$180(10.6556) = \$1918.01$$

Problems

1. Find the present value of a pension of \$75 a month payable at the end of each 3 months to a pensioner aged 65.

2. A corporation executive aged 58 is to be retired at age 65. During retirement he will receive \$3,600 a year payable in monthly installments. Find the present value of this retirement allowance on a $3\frac{1}{2}\%$ basis.

3. A life annuity contract provided for a payment of \$750 a year for 15 years, the first payment to be made at age 60. At age 60 the annuitant desired monthly payments. Find the amount of the monthly payments.

4. A widow was to receive \$1,800 a year for life in annual payments, the first payment to be made one year after her husband's death. When the first payment was due, the widow was 65 and asked that the payments be made monthly. What amount should she receive monthly?

Forborne temporary annuity due. A forborne temporary annuity due is created when a person who is entitled to a life annuity due of 1 a year forbears to draw it and agrees that the unpaid installments are to accumulate as pure endowments until he is aged $x + n$.

On page 465 the present value of a temporary annuity due was found by the formula

$$\ddot{a}_{x\overline{n}|} = \frac{N_x - N_{x+n}}{D_x},$$

and on page 461 it is given that 1 at age x will purchase an n-year pure endowment of

$$\frac{D_x}{D_{x+n}}$$

Then the present value would buy a pure endowment equal to

$$\frac{D_x}{D_{x+n}} \times \frac{N_x - N_{x+n}}{D_x} \quad \text{or} \quad \frac{N_x - N_{x+n}}{D_{x+n}}$$

Problems

1. Find the amount at age 60 of a forborne temporary annuity due of 1 a year that is to be accumulated for a person now aged 40.

2. A man was to receive a life annuity of \$1200 a year, the first payment to be made one year after his 60th birthday. At that time he was still employed at a good salary, and so decided to postpone the beginning of the annuity for 5 years. What yearly sum will he receive, the first payment to begin on his 66th birthday? (Use American Experience Table of Mortality and $3\frac{1}{2}\%$, and treat the postponed annuity as a forborne annuity.)

CHAPTER 40

Net Premiums

Net single premium. The net single premium is equal to the present value of the benefit influenced by rates of mortality and interest.

The net single premium for a whole life policy (a policy payable at death only) is denoted by A_x. Solution of a problem of this type is simplified by use of the commutation columns M_x and D_x, thus:

$$A_x = \frac{M_x}{D_x}$$

Example

Find the net single premium for \$3,000 whole life insurance on a person aged 24.

Solution

$$A_{24} = \frac{M_{24}}{D_{24}}$$

$$= \frac{11935.4}{39307.1} = .303644$$

$$\$3{,}000 \times .303644 = \$910.93$$

Annual premiums. Life insurance premiums are most frequently paid in equal annual payments, but they may be paid semiannually, quarterly, or monthly, and, in the case of industrial insurance, weekly. Rates other than annual are greater in proportion than annual rates, for they include interest and additional overhead or administrative costs.

On an ordinary life policy, payments continue throughout the life of the insured. On a limited payment life policy, the premium payments are limited to a certain number of years, such as 20 years on a 20-payment life policy.

The net annual premium is the annual payment made at the beginning of each policy year, the sum thereof being the equivalent of the net single premium. The annual premiums constitute an annuity due payable by the policy holder to the insurance company. P_x is the symbol used for the net annual premium and, using commutation columns,

$$P_x = \frac{M_x}{N_x}$$

Example

Find the net annual premium for an ordinary life policy of $1,000 issued to a person aged 30.

Solution

$$P_{30} = \frac{M_{30}}{N_{30}} = \frac{10259.0}{596804} = .017189$$

$$\$1,000 \times .017189 = \$17.19$$

If the premium-paying period is limited to a certain number of years, such as 10 years in a 10-payment life policy, then the payments are equivalent, interest and mortality considered, to the single net premium. ${}_nP_x$ is the symbol used for the net annual premium for an n-payment life policy to a person aged x, and, in terms of commutation columns:

$${}_nP_x = \frac{M_x}{N_x - N_{x+n}}$$

Example

Find the net annual premium for a 20-payment life policy for $2000 issued to a person aged 45.

Solution

$${}_{20}P_{45} = \frac{M_{45}}{N_{45} - N_{65}} = \frac{7192.81}{253745 - 48616} = \frac{7192.81}{205129} = .03506$$

$$\$2000 \cdot .03506 = \$70.12$$

Term insurance. Other than group life insurance, term insurance is the lowest-cost life insurance obtainable. The term may be one year, five years, or ten years, and so forth, and the face value of the policy is payable in the event of death within the stated term.

The net single premium for term insurance may be ascertained from commutation columns, using the formula

$$A'_{x\overline{n}|} = \frac{M_x - M_{x+n}}{D_x}$$

Example

Find the net single premium for a 10-year term insurance of $5,000 at age 25

Solution

$$A'_{25\ \overline{10}|} = \frac{M_{25} - M_{35}}{D_{25}}$$

$$= \frac{11631.1 - 9094.96}{37673.6}$$

$$= \frac{2536.14}{37673.6} = .067318$$

$$\$5,000 \times .067318 = \$336.59$$

Annual premium for term insurance. The payments of premium constitute an annuity due for a definite term, and the net annual premium may be determined from commutation columns, using the formula

$$P'_{x\overline{n}|} = \frac{M_x - M_{x+n}}{N_x - N_{x+n}}$$

Example

Find the net annual premium for a 10-year term insurance of $3,000 at age 20.

Solution

$$\begin{aligned} P'_{20\,\overline{10}|} &= \frac{M_{20} - M_{30}}{N_{20} - N_{30}} \\ &= \frac{13267.3 - 10259.0}{984400 - 596804} \\ &= \frac{3008.3}{387596} = .007761 \end{aligned}$$

$$\$3{,}000 \times .007761 = \$23.38$$

Net single premium for endowment insurance. An endowment policy provides for payment of the face value of the policy at the end of the stated period if the insured be living, or to the named beneficiary or beneficiaries should death occur before the end of the stated period.

Endowment insurance may be considered as term insurance of 1 for n years plus an n-year pure endowment of 1, and the net single premium may be found from commutation columns by the use of the following formula:

$$A_{x\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}$$

Example

Find the net single premium on a 10-year endowment policy for $5,000 at age 30.

Solution

$$\begin{aligned} A_{30\,\overline{10}|} &= \frac{M_{30} - M_{40} + D_{40}}{D_{30}} \\ &= \frac{10259.0 - 8088.91 + 19727.4}{30440.8} \\ &= \frac{21897.49}{30440.8} = .719346 \end{aligned}$$

$$\$5{,}000 \times .719346 = \$3{,}596.73$$

Annual premium for endowment insurance. The net annual premium for r years for an n-year endowment insurance of 1 may be found from commutation columns, using the formula

$$P_{x\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}}$$

Example

Find the net annual premium on a 15-year endowment policy for $10,000 purchased at age 40.

Solution

$$P_{40\,\overline{15}|} = \frac{M_{40} - M_{55} + D_{55}}{N_{40} - N_{55}}$$

$$= \frac{8088.91 - 5510.54 + 9733.40}{344167 - 124876}$$

$$= \frac{12311.77}{219291} = .056143$$

$$\$10{,}000 \times .056143 = \$561.43$$

Miscellaneous Problems

1. Find the net single premium for a whole life insurance of $1,000 at age 30.

2. What is the increase in the net single premium for a whole life insurance of $2,000 from age 25 to 26?

3. What is the present value of a life annuity of $1,000 a year at age 25?

4. Find the net annual premium for an ordinary life policy of $1,000 at age 20.

5. Find the net annual premium for a 20-payment life policy for $10,000 at age 30.

6. What is the net annual premium for a 10-payment life policy for $2,000 at age 60?

7. What is the net single premium for 10-year term insurance of $5,000 at age 20?

8. What is the net annual premium on a 10-year endowment policy for $5,000 at age 45?

9. Find the net annual premium for a 20-payment endowment at age 65 for $5,000 if the insured is 45 at date of issue.

10. Find the difference in annual premiums between a 20-payment life policy and a 20-year endowment policy, each for $5,000, issued at age 30.

CHAPTER 41

Valuation of Life Insurance Policies

Mortality and the level premium. If the cost of insurance on a group of men were to be met each year by payment into a fund of just the amount necessary to meet that year's death loss, the amount would be low at first, and finally prohibitive, hence the necessity of a level premium. In order to have a level premium, an excess over current death losses is collected during the early years to bear the burden of later years when losses exceed the premium income.

This excess premium is known as the "reserve." When calculated on the basis prescribed by law, it is called the "legal reserve," and "legal reserve" companies are referred to as "old line" companies.

Policy reserves. To show the meaning of insurance reserves, a simple illustration is given.

Assume that an ordinary life policy for $1,000 is purchased at age 20. The net annual premium, calculated from Table 8, would be

$$P_{20} = \frac{M_{20}}{N_{20}} = \frac{13267.3}{984400} = .013487$$

$$\$1{,}000 \times .013847 = \$13.85$$

Term insurance for one year at the same age would be:

1st year:

$$A_{20} = \frac{C_{20}}{D_{20}} = \frac{351.07}{46556.2} = .00754$$

$$\$1{,}000 \times .00754 = \$7.54$$

5th year:

$$A_{25} = \frac{C_{25}}{D_{25}} = \frac{293.55}{37673.6} = .00779$$

$$\$1{,}000 \times .00779 = \$7.79$$

and so on.

Comparing these premiums over a period of years, we have:

Age	*Ordinary Life*	*One Year Term*
20	$13.85	$ 7.54
25	13.85	7.79
30	13 85	8.14
35	13 85	8 64
40	13 85	9.46
45	13 85	10 79
50	13 85	13 32
55	13 85	17 94
60	13 85	25 79
65	13 85	38.77

The excess of the premium on ordinary life over one-year term insurance is the amount placed in the reserve to be accumulated for heavier losses which will occur. It will be noticed that between 50 and 55 and from that point on the ordinary life premiums will be insufficient; therefore, the reserves will be drawn upon to meet the difference.

Interest and the premium. Reserves are invested in securities and earn interest which increases the reserves. The amount of interest earned, therefore, is reflected in lower premiums. If a company assumes a rate of interest lower than the maximum permitted under insurance law, the premium is higher and a larger reserve is accumulated during the early policy years.

Loading. Using the mortality table and an assumed rate of interest to be earned on the reserve, the actuary arrives at the net level premium. To this must be added the expense of doing business, or overhead, which amount is called "loading." The net premium plus the loading is the premium rate to the purchaser of insurance.

Expenses are heaviest in the first policy year; therefore, the plan is modified to permit more of the premium to be used for expenses, and this is balanced by lowering the amount required for the reserve. Methods of modification* are not presented in this text.

Dividends and net cost. Mortality may differ from that shown by the table; interest may be earned in excess of the rate anticipated; the loading may exceed the actual costs. Such savings result in dividends to policy holders in mutual companies and to holders of participating policies issued by stock companies. The net cost of the insurance is the premium paid less these dividend refunds.

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* An extended discussion of these methods will be found in Robert Riegel and H. J. Loman, *Insurance Principles and Practice.* New York: Prentice-Hall, Inc., rev. ed., 1929.

Terminal reserves. When a policy is issued, the mathematical expectation of the future premiums equals the benefit.

As the insured grows older, the value of the future premiums becomes less and the value of the benefit, conversely, becomes greater.

The value of the benefit is represented by A_{x+n}, and the net annual premium constitutes a life annuity with a value represented by $P_x(1 + a_{x+n})$; therefore, ${}_nV_x$, the terminal reserve, is found by the formula

$$ {}_nV_x = A_{x+n} - P_x(1 + a_{x+n}) $$

The foregoing method of valuation is called the *prospective* method.

Example

Find the terminal reserve of the 20th policy year on an ordinary life policy of $1,000 issued at age 30.

Solution

$$ {}_{20}V_{30} = A_{50} - P_{30}(1 + a_{50}) $$

$$ A_{50} = \frac{M_{50}}{D_{50}} = \frac{6355.44}{12498.6} = .508492 $$

$$ a_{50} = \frac{N_{51}}{D_{50}} = \frac{169165}{12498.6} = 13.5347 $$

$$ 1 + 13.5347 = \$14.5347 $$

$$ P_{30} = \frac{M_{30}}{N_{30}} = \frac{10259.0}{596804} = .017189 $$

Substituting values gives:

$$ {}_{20}V_{30} = .50849 - (.017190 \times 14.5347) $$
$$ = .25864 $$
$$ \$1,000 \times .25864 = \$258.64 $$

The surrender value is the sum which the insurance company pays the policy holder upon the surrender and cancellation of the policy. Whether the amount will be greater or less than an amount as calculated in the foregoing example is dependent on averages obtained from the company's records instead of the theoretical amount so calculated. Policies contain a table showing the company's contractual surrender value for each $1,000 of insurance.

Problems

1. Find the terminal reserve of the tenth year on an ordinary life policy of $3,000 taken at age 20.

2. Find the terminal reserve of the twentieth year on an ordinary life policy of $5,000 taken at age 30.

Retrospective method. Under this method the policy value is found by deducting the accumulated losses from the accumulated premiums. The formula is:

$$_nV_x = \frac{M_x}{N_x} \cdot \frac{N_x - N_{x+n}}{D_{x+n}} - \frac{M_x - M_{x+n}}{D_{x+n}}$$

Problems

1. Check the answer to the example, using the retrospective formula.

2. Find the surrender value in Problems 1 and 2 on page 475 by the retrospective method.

Transformation. In computing the terminal reserve, the formula used was

$$_nV_x = A_{x+n} - P_x(1 + a_{x+n})$$

and the net single premium at age x, denoted by A_x, was computed by the formula:

$$A_x = P_x(1 + a_x)$$

Then, substituting for A_x, we have:

$$\begin{aligned} _nV_x &= P_{x+n}(1 + a_{x+n}) - P_x(1 + a_{x+n}) \\ &= (P_{x+n} - P_x)(1 + a_{x+n}) \end{aligned}$$

which represents the policy value or reserve, for the policy value is equal to the present value of the difference between the net premiums for age $x + n$ and age x for the remainder of life.

To express the value of the reserve in terms of annuities, take the formula:

$$A_x = 1 - d(1 + a_x)$$

which denotes the present value of 1 payable at the end of the year in which a person dies (d being the value, at the beginning of the year, of the interest for each year on 1), and

$$P_x = \frac{1}{1 + a_x} - d$$

which denotes the annual premium P_x, for an ordinary life policy expressed in terms of annuity values. Then substitute the values of A_x and P_x in the formula:

$$_nV_x = A_{x+n} - P_x(1 + a_{x+n})$$

Following the algebraic processes of simplifying, the result is

$$_nV_x = 1 - \frac{1 + a_{x+n}}{1 + a_x}$$

Problems

1. With the aid of the table of life annuities, calculate the terminal reserve of the twentieth policy year on an ordinary life policy for $1,000 issued at age 30.

2. Find the terminal reserve the tenth year on an ordinary life policy of $5,000 issued at age 25.

Reserve valuation for limited payment life insurance. In determining the reserve valuation for such policies as 10-payment life, 20-payment life, and so forth, the following principle is fundamental: the terminal reserve of the nth policy year equals the net single premium at the attained age of the insured minus the present value of the future net premiums. Using m to denote the number of annual payments, we have

$$n:{}_mV_x = A_{x+n} - {}_mP_x(1 + a_{x+n\ \overline{m-n-1}|})$$

but only when n is less than m. Where n is equal to or greater than m, the terminal reserve is simply equal to the net single premium.

For endowment insurance, the formula is

$${}_nVE_{x\overline{r}|} = AE_{x+n\ \overline{r-n}|} - PE_{x\overline{r}|}(1 + a_{x+n:\overline{r-n-1}|})$$

for an r-year endowment.

Problems

1. Find the terminal reserve of the fifteenth policy year for a $2000 20-year pay-life policy issued at age 30.

2. Find the terminal reserve of the fifteenth policy year for a $3000 20-year endowment issued at age 45.

Preliminary term valuation. Initial expenses of securing a policy, such as agents' commissions, medical and inspection fees, and other expenses, make it practically impossible to provide any reserve out of the first year's premium. Under the net level premium method, the loading is the same each year; therefore, expenses exceed income, and the deficit must be met from general funds. To avoid this, the preliminary term valuation is used, whereby the first year's premium becomes available for expenses and losses, the policy is renewed at the beginning of the second year, and policy values begin with that year. Therefore, the first year is simply term insurance.

Under this plan, assume an ordinary life policy of $1,000 at age 20, the premium being $16.50.

The net premium for the first year would be:

$$A^1_{20\ \overline{1}|} = \frac{C_{20}}{D_{20}} = \frac{351.07}{46556.2} = .00754$$

$1,000 × .00754 = $7.54

$16.50 − $7.54 = $8.96, the loading for the first year.

For the second and subsequent years, the net premium will be the level net premium based upon age 21:

$$P_{21} = \frac{M_{21}}{N_{21}} = \frac{12916.3}{937843} = .01377$$

$\$1,000 \times .01377 = \$13.77$

$\$16.50 - 13.77 = \$2.73$, the loading for each year after the first.

Problems

1. Find the loading for the first and second years on an ordinary life policy of \$2000 at age 30, with an annual premium of \$42.60, using the preliminary term evaluation.

2. Using the preliminary term evaluation, find the loading for the first and second years on an ordinary life policy of \$1000 at age 40, the premium being \$28.80.

APPENDIXES

APPENDIX I

Practical Business Measurements

Practical business measurements. The measurements discussed in this section are those of practical use, and include measurements of or pertaining to the following: angles; surfaces; triangles, rectangles, and other polygons; circles, including circumference, radius, diameter, and area; area of irregular figures; and solids, such as the sphere, cone, cylinder, cube, and prismatoid.

Rectilinear figures. An angle is the difference in the direction of two lines proceeding from a common point called the vertex.

A right angle is an angle formed by two lines perpendicular to each other, and is an angle of 90°.

An angle that is less than a right angle is an acute angle, and one that is greater is an obtuse angle. Acute and obtuse angles are also called oblique angles.

A surface has two dimensions—length and breadth.

A plane, or a plane surface, is a level surface. A straight edge will fit on it in any position.

A plane figure is a figure all of whose points lie in the same plane.

A quadrilateral is a plane figure bounded by four straight lines.

Quadrilaterals are of three classes or kinds: the trapezium, which has four unequal sides, no two of which are parallel; the trapezoid, which has two and only two sides parallel; and the parallelogram, which has two pairs of parallel sides.

The altitude of a quadrilateral having two parallel sides is the perpendicular distance between those sides.

The diagonal of a quadrilateral is the straight line connecting two of its opposite vertices.

Parallelograms are of three classes or kinds: the rhomboid, which has one pair of parallel sides greater in length than the other pair, and no right angles; the rhombus, all of whose four sides are equal; and the rectangle, whose angles are all right angles.

A square is a rectangle having four equal sides. It is also a rhombus whose four angles are each 90°.

A triangle is a plane figure bounded by three straight lines. If the three sides are of equal length, the triangle is called equilateral. If two sides are of equal length, it is called isosceles. If the three sides are of different lengths, it is called scalene. If one of the three angles is a right angle, the triangle is called a right triangle, and the side opposite the right angle is called the hypotenuse.

Plane figures may be regular or irregular. A regular plane figure has all its sides and all its angles equal The smallest regular plane figure is

an equilateral triangle; the next, a square; the next, a pentagon; and so on. Each figure derives its name from the number of its angles or sides—hexagon, heptagon, octagon, nonagon, decagon, etc.

The perimeter of a plane figure is the sum of the lengths of its sides

The apothem of a polygon is a perpendicular line drawn from the center of the figure to the middle of a side, the center being the point within the figure which is equally distant from the middle points of all the sides.

The altitude of a plane figure is the perpendicular distance from the highest point above the base to the base or to the base extended.

Circles. A circle is a plane figure bounded by a curved line, called the circumference, every point of which is equally distant from a point within called the center.

The diameter of a circle is a straight line drawn through the center and terminated by the circumference.

The radius of a circle is a straight line drawn from the center to the circumference, and is equal to one-half the diameter.

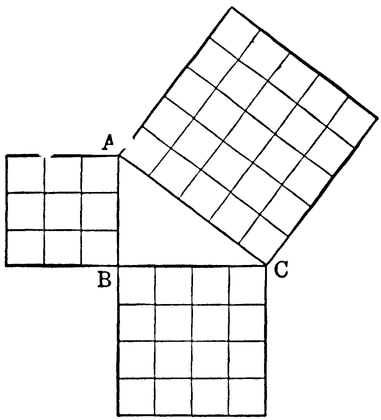


Figure 16.

An arc of a circle is any portion of the circumference.

A sector of a circle is bounded by two radii and the intercepted arc.

A chord is the straight line joining the extremities of an arc.

A segment is bounded by an arc and its chord.

A zone is a portion of a circle bounded by two parallel chords.

A tangent to a circle is a straight line having only one point in common with the curve; it simply touches the circle. A secant enters the figure from without.

An ellipse is a plane figure bounded by an oval curved line, and has a long and a short diameter or axis.

Measurement of triangles. It is proved in geometry that the square erected on the hypotenuse of a right triangle is equal to the sum of the squares erected on the other two sides. This may be illustrated as in Figure 16.

To find the length of the hypotenuse of a right triangle, the lengths of the other two sides being given, add the squares of the sides forming the right angle, extract the square root of the sum, and the result will be the length of the hypotenuse.

To find the length of either of the two sides other than the hypotenuse, from the square of the hypotenuse subtract the square of the given side, extract the square root of the remainder, and the result will be the length of the third side.

To find the area of a triangle, the base and the altitude being given, multiply the base by one-half the altitude.

To find the area of a triangle when the lengths of the three sides are given, from half the sum of the three sides, subtract the length of each side separately. Find the continued product of the three remainders and the half sum. The square root of the result will be the area.

Measurement of rectangles. The area of a square or of a rectangle is the product of the length and the breadth.

Either dimension of a rectangle may be found by dividing the area by the given dimension.

Measurement of quadrilaterals. The area of a trapezium may be found by multiplying one-half the sum of the altitudes by the diagonal.

The area of a trapezoid may be found by multiplying the sum of the parallel sides by one-half the altitude.

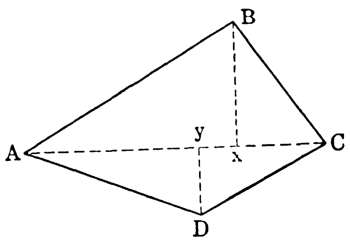


Figure 17. Trapezium.

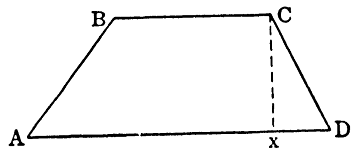


Figure 18. Trapezoid.

The area of a parallelogram may be found by multiplying the base by the altitude.

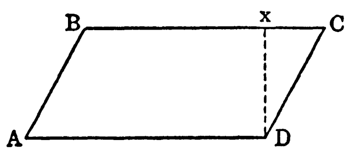


Figure 19. Parallelogram.

The area of polygons having equal sides and equal angles may be found by multiplying the square by one of the equal sides by:

.433, if the figure is a triangle
1.7205, if the figure is a pentagon
2.5981, if the figure is a hexagon
4.8284, if the figure is an octagon

Measurement of circles. It is shown in geometry that the circumference of a circle bears a fixed ratio to its diameter. This constant ratio is represented by π (pronounced "pi"), and is 3.1416.

From this relation the following principles are derived:

The circumference = the diameter × 3.1416
The diameter = the circumference ÷ 3.1416
The area = the circumference × half the radius

The area of a circle is found by considering the surface to be composed of an infinite number of isosceles triangles, the bases of which, taken together, equal the perimeter of the circle. The common altitude of these triangles constantly approaches the radius of the circle, and will reach that length when the perimeter consists of very short straight lines; hence, perimeter (circumference) × $\frac{1}{2}$ radius = area.

To find the circumference of a circle, multiply the diameter by 3.1416, or divide the area by one-fourth of the diameter.

To find the diameter of a circle, divide the circumference by 3.1416, or divide the area by .7854 and extract the square root of the result.

To find the area of a circle, multiply the circumference by one-half the radius; or, multiply the diameter by one-fourth of the circumference; or, multiply the square of the diameter by .7854; or, multiply the square of the radius by 3.1416.

To find the area of an ellipse, multiply the major axis by the minor axis, and that result by .7854.

To find the area of a sector of a circle, multiply one-half the length of the arc by the radius; or, take the same part of the area of the circle as the number of degrees in the arc is of 360°.

To find the area of a segment which is less than a semi-circle, from the area of a corresponding sector, subtract the area of the triangle formed by the chord and radii; to find the area of a segment which is greater than a semi-circle, add the area of the triangle formed by the chord and radii to the area of a corresponding sector.

To find the area of a zone, from the area of the circle subtract the areas of the segments not included in the zone.

Problems

1. Harry and George start from the same point, Harry going 4 miles due west, and George 3 miles due north; how far apart are they?

2. The base of a triangle is 12 inches, and the altitude is 8 inches. What is the area of the triangle?

3. The three sides of a triangular plot of land are 100 feet, 130 feet, and 150 feet. What is the area of the plot?

4. A rectangular piece of land is 40 rods long and 20 rods wide. What is the area in square rods?

5. Find the cost of fencing a field 40 rods wide and 55 rods long, if the fencing costs $2.25 a rod.

6. A field is in the form of a trapezium, having a diagonal of 90 rods, and altitudes of 25 rods and 40 rods. What is the area in square rods?

7. One of the parallel sides of a garden is 60 yards long, and the other is 80 yards long. The garden is 52 yards wide. How many square yards does it contain?

8. Find the area of a parallelogram whose base is 10 feet, and whose altitude is 4 feet.

9. The side of a hexagonal building is 20 feet. What is the floor area?

10. A cylindrical tank is 12 feet in diameter. What must be the length of a piece of strap iron which is to be used to make a band around the tank, if 1 foot is allowed for overlapping?

11. The circumference of a circle is 44 feet. What is its diameter?

12. Find the area of the circle in problem 11.

13. The diameters of an ellipse are 60 feet and 40 feet. What is the area?

14. How much belting will be required to make a belt to run over two pulleys, each 30 inches in diameter, if the distance between the centers of the pulleys is 18 feet?

15. If there is a steam pressure of 90 pounds to the square inch, what is the pressure on a 9-inch piston?

16. If pieces of sod are 12 inches by 14 inches, how many pieces will be required to sod a lawn 24 feet wide and 28 feet long?

17. Find the cost of painting the four side walls of a room 14 feet long, 10 feet 6 inches wide, and 8 feet high, at 18 cents a square yard, no allowance being made for openings.

18. A circular walk 5 feet wide is laid around a plot 20 feet in diameter. What is the cost of the walk at $2.50 a square foot?

19. Find the number of paving blocks required to pave a street one mile long and 35 feet wide, if the blocks are one foot long and five inches wide.

20. What part of an acre is a plot of land 78 feet long and 36 feet wide?

Solids. A solid is a magnitude which has length, breadth, and thickness. Solids include the prism, the cylinder, the pyramid, the cone, the polyhedron, and the sphere.

A prism is a solid whose upper and lower bases are equal and parallel polygons, and whose sides, or lateral faces, are parallelograms.

A rectangular sold is bounded by six rectangular surfaces.

A cube is a rectangular solid having six square faces.

A triangular prism is a prism whose bases are triangles.

A cylinder is a prism having an infinite number of faces or sides; the two bases are equal parallel circles.

A pyramid is a solid having for its base a polygon, and for its other faces three or more triangles which terminate in a common point called the vertex or apex.

A cone is a pyramid having an infinite number of faces; or, it is a solid whose base is a circle, and whose convex surface tapers uniformly to a point called the apex.

A polyhedron is a solid bounded by four or more faces.

A sphere is a solid bounded by a curved surface, every point of which is equally distant from a point within, called the center.

The frustum of a pyramid or of a cone is the solid which remains when a portion which includes the apex is cut off by a plane parallel to the base.

The axis of a pyramid or of a cone is a straight line that joins the apex to the center of the base.

The altitude of a pyramid or of a cone is the perpendicular height from its apex to its base.

The slant height of a pyramid is the distance from the apex to the midpoint of one side of its base.

The slant height of a cone is the distance from its apex to the circumference of its base.

The diameter of a sphere is a straight line drawn through its center and terminated at both ends by the surface.

The radius of a sphere is one-half of its diameter.

The circumference of a sphere is the greatest distance around the sphere.

A hemisphere is one-half of a sphere.

Measurement of solids. To find the contents of a prism or of a cylinder when the perimeter of the base and the altitude are given, multiply the area of the base by the altitude.

To find the convex surface of a prism or of a cylinder, multiply the perimeter of the base by the height.

To find the entire surface of a prism or of a cylinder, add the area of the bases to the area of the convex surface.

To find the convex surface of a cone, multiply the circumference of the base by one-half the slant height.

To find the entire surface of a cone, add the area of the base to the area of the convex surface.

The slant height of a pyramid or of a cone may be found by adding the square of the altitude to the square of the radius of the base, and extracting the square root of the sum.

To find the volume of a pyramid or of a cone, multiply the area of the base by one-third the altitude.

The volume of a pyramid is one-third as much as the volume of a prism that has the same base and altitude.

The volume of a cone is one-third as much as the volume of a cylinder that has the same base and altitude.

To find the convex surface of a frustum of a pyramid or of a cone, multiply one-half the sum of the perimeters of the two bases by the slant height.

To find the entire surface of a frustum of a pyramid or of a cone, add the area of the two bases to the area of the convex surface.

To find the volume of a frustum of a pyramid or of a cone, find the product of the areas of the two bases, and extract the square root thereof This result is the area of a base which is a mean base between the other two Add the three areas, and multiply by one-third the altitude.

To find the surface of a sphere, find the area of a great circle of the sphere, and multiply this area by 4.

To find the volume of a sphere, multiply the convex surface by one-third the radius.

The volume of a spherical shell (a hollow sphere) is equal to the volume of the outside sphere minus the volume of the inside sphere.

Problems

1. A cylindrical tank is 12 feet in diameter. If it is filled with water to a depth of 6 feet, what is the weight of the water? (1 cu. ft. of water weighs 62.5 pounds.)

2. How many square yards of sheet metal will be required for a smoke-stack 2 feet in diameter and 12 feet in height, if 1 inch is allowed for overlapping?

3. Find the cost of painting the entire surface of a cylindrical tank 10 feet in diameter and 20 feet long, at 10¢ per square foot.

4. The boundary lines of the Fort Pembina Airport are marked by cone-shaped markers; each marker is 3 feet in diameter and has a slant height of 3 feet. If there are 120 of these markers, and 1 inch was allowed for overlapping, how many square feet of sheet metal were required for their construction?

5. If a freight car is 36 feet long, and 8 feet 6 inches wide, inside measure, how many bushels of wheat will it contain when filled to a depth of 5 feet? (A cubic foot is approximately .8 of a bushel.)

6. How many tons of coal will fill a bin 20 feet, by 16 feet, by 8 feet, if there are 80 cubic feet to a ton?

7. The measurements of a railroad embankment are: length, 400 feet; height, 10 feet; width of base, 14 feet; and width of top, 8 feet. How many cubic yards of earth will be required?

8. The measurements of a funnel are as follows: larger diameter, 12 inches; smaller diameter, 1 inch; and slant height, 18 inches. How many square inches of sheet metal will be required?

9. One of the units of a grain elevator is a concrete cylinder 20 feet in diameter and 50 feet in height. The bottom is cone-shaped to facilitate the drawing off of the grain. The depth of this cone is 5 feet. If the wheat in this unit is leveled off at the 30-foot mark, how many bushels are in the unit, assuming that a cubic foot is approximately .8 of a bushel?

10. A bucket is 16 inches wide at the top, and 10 inches wide at the bottom. The depth is 12 inches. How many gallons of water will the bucket hold? (231 cu. in. = 1 gal.)

11. What number of square feet of sheet metal will be required to make 100 pails, each 10 inches deep, 8 inches in diameter at the bottom, and 11 inches in diameter at the top? The allowance for seams and for waste in cutting is 10%.

12. How many tiles 1 inch square will be required for the surface of a tiled dome in the form of a hemispherical surface, if the diameter of the dome is 24 feet?

13. The top of a vat is 9 feet square, and the base is 8 feet square. If the slant height is 10 feet, what is the capacity of the vat in cubic feet?

14. How many cubic feet are there in a spherical body whose diameter is 10 feet?

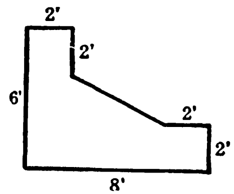
15. The base of a church steeple is in the form of an octagon measuring 6 feet on each side. The slant height of the steeple is 80 feet. What will be the cost of painting this steeple at 50¢ per square yard?

16. A tank is 8 feet long, 6 feet wide, and 3 feet deep. If a cubic foot of water weighs 62.5 pounds, what is the weight of water in this tank if it is two-thirds full?

17. If 38 cubic feet of coal weigh a ton, how many tons can be put into a bin 10 feet long and 8 feet wide, if the coal is leveled off at an average depth of 5 feet?

18. Find the number of square feet of sheet metal required to make 12 gross of pails, each 14 inches deep, 8 inches in diameter at the bottom, and 11 inches in diameter at the top, not allowing for seams or waste in cutting.

19. The diagram is that of a cross section of a concrete retaining wall 150 feet long. Find the number of cubic yards of material necessary to construct such a wall.



20. If 200 gallons of water flow through a pipe 2 inches in diameter in 4 hours, how much water will flow through a pipe 4 inches in diameter in the same time? Hint: The amounts of the liquids are to each other as the squares of the like dimensions.

21. A cylindrical hot water tank is 5 feet high and 11 inches in diameter. How many gallons will it contain?

22. A corn crib 32 feet by 10 feet by 8 feet is filled with ear corn. How many bushels will it contain if one bushel equals $1\frac{1}{2}$ cubic feet?

23. A barn loft is 36 feet by 24 feet by 8 feet. How many tons of hay will it hold if it is to be filled with: (*a*) clover hay weighing one ton for 600 cubic feet; (*b*) timothy hay weighing one ton for 500 cubic feet?

24. If a heaped bushel equals $1\frac{1}{4}$ cubic feet, how many bushels of potatoes may be stored in a bin that is 12 feet by 8 feet by 6 feet?

25. If a cubic foot of steel weighs 484 pounds, what is the weight of a hollow steel cylinder whose length is 10 feet and the radii of whose outer and inner circles are 3 feet and $2\frac{1}{2}$ feet, respectively?

APPENDIX II

Tables of Weights, Measures, and Values

Long Measure

U. S. and British Standard

12 inches	1 foot
3 feet	1 yard
$5\frac{1}{2}$ yards, or $16\frac{1}{2}$ feet	1 rod
320 rods, or 5,280 feet	1 mile
1,760 yards	1 mile
40 rods	1 furlong
8 furlongs	1 statute mile
3 miles	1 league

Metric System

10 millimeters	1 centimeter
10 centimeters	1 decimeter
10 decimeters	1 meter
10 meters	1 dekameter
10 dekameters	1 hektometer
10 hektometers	1 kilometer
10 kilometers	1 myriameter

Comparisons of Long Measures

1 inch	25.4001 millimeters	1 centimeter	.3937 inch
1 foot	.304801 meter	1 meter	39.37 inches
1 yard	.914402 meter	1 meter	3.28083 feet
1 rod	5.029 meters	1 meter	1.093611 yards
1 mile	1.60935 kilometers	1 kilometer	.62137 mile

Square Measure

U. S. and British Standard

144 square inches	1 square foot
9 square feet	1 square yard
$30\frac{1}{4}$ square yards	1 square rod
$272\frac{1}{4}$ square feet	1 square rod
40 square rods	1 rood
4 roods	1 acre
160 square rods	1 acre
640 acres	1 square mile
43,560 square feet	1 acre
4,840 square yards	1 acre

Metric System

100 square millimeters	1 square centimeter
100 square centimeters	1 square decimeter
100 square decimeters	1 square meter
100 square meters	1 square dekameter
100 square dekameters	1 square hektometer
100 square hektometers	1 square kilometer
100 square kilometers	1 square myriameter

Comparisons of Square Measures

1 sq. in.	6.452 sq. cm.	1 sq. mm.	.00155 sq. in.
1 sq. ft.	.0929 sq. m.	1 sq. cm.	.155 sq. in.
1 sq. yd.	.8361 sq. m.	1 sq. m	10.764 sq. ft.
1 sq. rd.	25.293 sq. m.	1 sq. m.	1.196 sq. yds.
1 sq. mi.	2.59 sq. km.	1 sq. km.	.3861 sq. mi.
		1 sq. km	247.11 acres
		1 sq. Dm., or 1 are	1,076.41 sq. ft.
		100 ares = 1 hektare	2.4711 acres

Solid or Cubic Measure (Volume)

U. S. and British Standard		*Metric System*	
1,728 cubic inches	1 cubic foot	1,000 cubic millimeters	1 cu. cm.
27 cubic feet	1 cubic yard	1,000 cubic centimeters	1 cu. dm.
128 cubic feet	1 cord of wood	1,000 cubic decimeters	1 cu. m.
$24\frac{3}{4}$ cubic feet	1 perch of stone	1,000 cubic meters	1 cu. Dm.
2,150.42 cubic inches	1 standard bushel	1,000 cubic dekameters	1 cu. Hm.
231 cubic inches	1 standard gallon	1,000 cubic hektometers	1 cu. Km.
40 cubic feet	1 ton (shipping)	1,000 cubic kilometers	1 cu. Mm.

Comparisons of Solid or Cubic Measures (Volume)

1 cu. in.	16.3872 cu. cm.	1 cu. cm.	.061 cu. in.
1 cu. ft	.02832 cu. m.	1 cu. m.	35.314 cu. ft.
1 cu. yd.	.7646 cu. m.	1 cu. m.	1.3079 cu. yds.
		1 cu. dm. = 1 liter	61.023 cu. in.
		1 liter	1.05671 liquid quarts
		1 liter	.9081 dry quart
		1 hectoliter or decistere	3.5314 cu. ft. or 2.8375 U. S. Bushels
		1 stere, kiloliter, or cu. m	1.3079 cu. yds. or 28.37 U. S. Bushels

Liquid Measure (Capacity)

U. S. and British Standard		*Metric System*	
4 gills	1 pint	10 milliliters	1 centiliter
2 pints	1 quart	10 centiliters	1 deciliter
4 quarts	1 gallon	10 deciliters	1 liter
$31\frac{1}{2}$ gallons	1 barrel	10 liters	1 dekaliter
2 barrels	1 hogshead	10 dekaliters	1 hektoliter
1 U. S. Gallon	231 cubic inches	10 hektoliters	1 kiloliter
1 British Imperial Gallon	277.274 cubic inches	10 kiloliters	1 myrialiter
7.4805 U. S. Gallons	1 cubic foot		
16 fluid ounces	1 pint		
1 fluid ounce	1.805 cubic inches		
1 fluid ounce	29.59 cubic centimeters		
1.2 U. S. Quarts	1 Imperial Quart		
1.2 U. S. Gallons	1 Imperial Gallon		
1 gallon gasoline	6 pounds (approx.)		
1 gallon oil	$7\frac{1}{2}$ pounds (approx.)		
1 gallon water	8.3 pounds (approx.)		
1 liter gasoline	1.59 pounds (approx.)		
1 liter gasoline	0.72 kilograms		

Dry Measure

U. S. and British Standard		*Metric System*
2 pints	1 quart	
8 quarts	1 peck	
4 pecks	1 bushel	[In the Metric System,
2,150.42 cubic inches	1 U. S. Standard Bushel	the same table is used
1.2445 cubic feet	1 U. S. Standard Bushel	for both Liquid Meas-
2,218.192 cubic inches	1 British Imperial Bushel	ure and Dry Measure.]
1.2837 cubic feet	1 British Imperial Bushel	

Comparisons of Liquid and Dry Measures

1 liquid quart	.94636 liter
1 liquid gallon	3.78543 liters
1 dry quart	1.1012 liters
1 peck	8.80982 liters
1 bushel	.35239 hektoliters
1 milliliter	.03381 liquid ounce, or .2705 apothecaries' dram
1 liter = 1 cubic decimeter	61.023 cubic inches .03531 cubic foot .2642 U. S. Gallon 2.202 pounds of water at 62° F.
28.317 liters	1 cu. ft.
4.543 liters	1 British Imperial Gal.
3.785 liters	1 U. S. Gal.

Avoirdupois Measure (Weight)

(Used for weighing all ordinary substances except precious metals, jewels, and drugs)

U. S. and British Standard		*Metric System*	
$27\frac{11}{32}$ grains	1 dram	10 milligrams	1 centigram
16 drams	1 ounce	10 centigrams	1 decigram
16 ounces	1 pound	10 decigrams	1 gram
25 pounds	1 quarter	10 grams	1 dekagram
4 quarters	1 hundredweight	10 dekagrams	1 hektogram
100 pounds	1 hundredweight	10 hektograms	1 kilogram
20 hundredweight	1 ton	10 kilograms	1 myriagram
2,000 pounds	1 short ton		
2,240 pounds	1 long ton		

Troy Measure (Weight)

(Used for weighing gold, silver, and jewels)

24 grains	1 pennyweight
20 pennyweights	1 ounce
12 ounces	1 pound

Apothecaries' Measure (Weight)

(Used for weighing drugs)

20 grains	1 scruple
3 scruples	1 dram
8 drams	1 ounce
12 ounces	1 pound

Comparison of Avoirdupois and Troy Measures

1 pound troy	5,760 grains	1 ounce troy	480 grains
1 pound avoirdupois	7,000 grains	1 ounce avoirdupois	$437\frac{1}{2}$ grains

1 karat, or carat	3.2 troy grains
24 karats	pure gold

Comparison of Avoirdupois and Troy Measures with Metric Weights

1 grain	.0648 gram
1 ounce (avoir.)	28.3495 grams
1 ounce (troy)	31.10348 grams
1 pound (avoir.)	.45359 kilogram
1 pound (troy)	.37324 kilogram

1 gram	15.4324 grains .03527 ounce (avoir.) .03215 ounce (troy)
1 kilogram	2.20462 pounds (avoir.) 2.67923 pounds (troy)
1 tonne, or metric ton	.9842 ton of 2,240 pounds, or 19.68 hundredweight 1.1023 tons of 2,000 pounds
1,000 kilograms	2,204.6 pounds
1.016 metric tons, or 1,016 kilograms	1 ton of 2,240 pounds

Apothecaries' Fluid Measure (Capacity)

60 minims	1 fluid dram
8 fluid drams	1 fluid ounce
16 fluid ounces	1 pint
8 pints	1 gallon

Comparisons (Approximate Liquid Measure)

Apothecaries'	*Common*	*Metric*
1 minim	1 to 2 drops	0.06 cu. cm.
60 minims, or 1 fluid dram	1 teaspoonful	3.75 cu. cm.
2 fluid drams	1 dessertspoonful	7.50 cu. cm.
4 fluid drams	1 tablespoonful	15.00 cu. cm.
8 fluid drams	1 fluid ounce	28.39 cu. cm.
2 fluid ounces	1 wineglassful	59.20 cu. cm.
4 fluid ounces	1 teacupful	118.40 cu. cm.
16 fluid ounces	1 pint	473.11 cu. cm

NOTE: Drops are not accurate measures, but for practical purposes it may be considered that one minim equals one drop of watery liquids and fixed oils, but two drops of volatile oils and alcoholic liquids, such as tinctures and fluid extracts.

MISCELLANEOUS TABLES

Surveyors' Long Measure

7.92 inches	1 link
25 links	1 rod
4 rods, or 100 links	1 chain
80 chains	1 mile

Surveyors' Square Measure

625 square links	1 square rod
16 square rods	1 square chain
10 square chains	1 acre
640 acres	1 square mile
36 square miles	1 township

Mariners' Measure

6 feet	1 fathom
120 fathoms	1 cable's length
$7\frac{1}{2}$ cable lengths	1 mile
5,280 feet	1 statute mile
6,080 feet	1 nautical mile, or British Admiralty knot
$50.71\frac{1}{15}$ feet	1 knot
120 knots, or $1.152\frac{2}{3}$ statute miles	1 nautical or geographical mile
3 geographical miles	1 league
60 geographical miles, or 69.16 statute miles	1 degree of longitude on the equator, or 1 degree of meridian
360 degrees	1 circumference

NOTE: A knot is properly $\frac{1}{120}$ of a marine mile, but current usage makes it equivalent to a marine mile. Hence, when the speed of vessels at sea is being measured, a knot is equal to a nautical mile, or 6,086.08 feet, or 2,028.69 yards.

Circular or Angular Measure

60 seconds (60″)	1 minute (1′)
60 minutes (60′)	1 degree (1°)
30 degrees	1 sign
90 degrees	1 right angle or quadrant
360 degrees	1 circumference

NOTE: One degree at the equator is approximately 60 nautical miles.

Counting

12 units or things	1 dozen
12 dozen, or 144 units	1 gross
12 gross	1 great gross
20 units	1 score

Paper Measure

24 sheets	1 quire
20 quires	1 ream
2 reams	1 bundle
5 bundles	1 bale

NOTE: Although a ream contains 480 sheets, 500 sheets are usually sold as a ream.

Books

Books are printed on large sheets of paper, which are folded into leaves according to the size of the book. The terms *folio, quarto, octavo,* and so forth, as applied to printed books, are based on sheets about 18 by 24 inches, or about half the size now generally used, and indicate the number of leaves into which each sheet is folded.

A sheet folded in	2 leaves is called	a folio	and makes	4 pages
" " " "	4 " " "	a quarto, or 4to	" "	8 "
" " " "	8 " " "	an octavo, or 8vo	" "	16 "
" " " "	12 " " "	a 12 mo	" "	24 "
" " " "	16 " " "	a 16 mo	" "	32 "
" " " "	24 " " "	a 24 mo	" "	48 "
" " " "	32 " " "	a 32 mo	" "	64 "

Sizes of Paper

Book Papers		*Bond, Ledger, and Writing Papers*	
25 × 38	38 × 50	14 × 17	20 × 28
$30\frac{1}{2}$ × 41	41 × 61	16 × 21	23 × 31
32 × 44	64 × 44	18 × 23	21 × 32
33 × 44	66 × 44	17 × 28	16 × 42
35 × 45	45 × 70	19 × 24	23 × 36
		17 × 22	22 × 34

MEASURES OF VALUE

United States Money

10 mills	1 cent
10 cents	1 dime
10 dimes	1 dollar
10 dollars	1 eagle

English Money

4 farthings	1 penny (d.)
12 pence	1 shilling (s.)
20 shillings	1 pound (£)

A pound sterling = $4.8665 (normal).

French Money

10 millimes (m.)	1 centime (c.)
10 centimes	1 decime (d.)
10 decimes	1 franc (fr.)

A franc = $0.193 (normal).

Comparison of Thermometer Scales

To convert from ° F to ° C, subtract 32 from ° F and divide by 1.8.
To convert from ° C to ° F multiply ° C by 1.8 and add 32.

Temperature Equivalents

Degrees C	*Degrees F*	*Remarks*
−100	−148	
− 50	− 58	
− 40	− 40	
− 20	− 4	
− 17 77	0	
− 15	5	
− 10	14	
− 5	23	
0	32	Water freezes
5	41	
10	50	
15	59	
20	68	
25	77	
30	86	
35	95	
40	104	
45	113	
50	122	
55	131	
60	140	
65	149	
70	158	
75	167	
80	176	
85	185	
90	194	
95	203	
100	212	Water boils at sea level
150	302	(With each 1,000 feet
190	374	altitude, boiling point of
200	392	water is reduced ap-
300	572	proximately 1° C.)

Approximate Weight of Substances

	Lbs. Per Cu. Ft.		*Lbs. Per Cu. Ft.*
Brick, pressed, best	150	Lead	709.6
Brick, common, hard	125	Limestone, marble, ordinarily	168
Brick, common, soft	100	Limestone, marble, piled	96
Coal, broken (anthra), loose	52–56	Masonry, granite, dressed	165
Coal, broken (bitu.), loose ...	47–52	Masonry, sandstone	145
Cement, concrete, limestone..	148	Sand, pure quartz dry loose	112–113
Cement, concrete, cinder	112	lbs. per struck bu	90–106
Cement, concrete, stone	150	Sand, angular, large and small.	117
Cement, concrete, trap rock	155	Sandstone, dry for building	151
Granite	170	Sandstone, quarried, piled ...	86
Hemlock, dry..	25	Shales, red or black...	162
Hickory, dry.	53	Shales, quarried, piled	92
Ice	57–0	Slate	175
Iron, cast...	450	Soapstone or steatite	170
Iron, wrought.	485	Steel, heaviest, lowest in carbon	490

WEIGHTS AND MEASURES

Solid Fuels

	Lbs. Per Cu. Yd.	*Tons Per Cu. Yd.*	*Cu. Ft. Per Ton*
Coal, anthracite egg	1514	.76	36
Coal, anthracite nut	1536	.77	36
Coal, anthracite stove	1521	.76	36
Coal, bituminous, Ill	1275	.64	42
Coal, bit., Ind. block	1161	.58	43
Coal, bit., Iowa lump	1256	63	42
Coal, bit., Pittsburgh	1255	.63	42
Coal, bit., Pocahontas egg and lump	1411	.71	38
Coal, cannel	1328	.66	49
Coke, loose	870–1026	51	60–65
Charcoal, hardwood	513	25	19
Charcoal, pine	486	.24	19
Peat, dry	1269	.63	42

Anthracite and Pocahontas, approximately 36 cu. ft. for 1 ton. Other bituminous coal, approximately $40\frac{1}{2}$ cu. ft. for 1 ton. Coke, approximately 60–65 cu. ft. for 1 ton.

Bulk Materials

	Lbs. Per Cu. Yd.	*Tons Per Cu. Yd.*
Ashes	1080	52
Asphalt	2700	1 35
Brick, soft clay	2718	1 35
Brick, hard clay	3397	1 69
Brick, pressed	3806	1.90
Bluestone	2970	1.48
Cement, Portland	2430	1.21
Cinders	1080	.54
Clay, dry	1701	85
Clay, wet	2970	1.48
Earth, dry, loose	1890	.94
Earth, dry, shaken	2214	1.10
Earth and sand, dry, loose	2700	1 35
Earth and sand, dry, rammed	3240	1.62
Fire brick	3915	1 95
Fire clay	3510	1 75
Gravel, dry	2970	1 48
Granite	4536	2.26
Lime, quick, shaken	1485	.70
Limestone, loose	2592	1.29
Marble, loose	2592	1.29
Mud, river	2430	1.21
Pitch	1863	.93
Rip-rap, limestone	2160	1.08
Rip-rap, sandstone	2430	1.21
Rip-rap, slate	2835	1.41
Sand, dry, loose	2619	1 30
Sand, wet	3186	1.55
Slag, screenings	2700	1.35
Street sweepings	850	.42
Tar	1674	.83

APPENDIX III—TABLES

Table 1
TABLE OF LOGARITHMS

N.	0	1	2	3	4	5	6	7	8	9	D
100	000000	000434	000868	001301	001734	002166	002598	003029	003461	003891	432
1	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
2	8600	9026	9451	9876	010300	010724	011147	011570	011993	012415	424
3	012837	013259	013680	014100	4521	4940	5360	5779	6197	6616	420
4	7033	7451	7868	8284	8700	9116	9532	9947	020361	020775	416
105	021189	021603	022016	022428	022841	023252	023664	024075	4486	4896	412
6	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
7	9384	9789	030195	030600	031004	031408	031812	032216	032619	033021	404
8	033424	033826	4227	4628	5029	5430	5830	6230	6629	7028	400
9	7426	7825	8223	8620	9017	9414	9811	040207	040602	040998	397
110	041393	041787	042182	042576	042969	043362	043755	044148	044540	044932	393
1	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	390
2	9218	9606	9993	050380	050766	051153	051538	051924	052309	052694	386
3	053078	053463	053846	4230	4613	4996	5378	5760	6142	6524	383
4	6905	7286	7666	8046	8426	8805	9185	9563	9942	060320	379
115	060698	061075	061452	061829	062206	062582	062958	063333	063709	4083	376
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7	8186	8557	8928	9298	9668	070038	070407	070776	071145	071514	370
8	071882	072250	072617	072985	073352	3718	4085	4451	4816	5182	366
9	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363
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6	100371	100715	101059	101403	101747	102091	102434	102777	103119	3462	343
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8	7210	7549	7888	8227	8565	8903	9241	9579	9916	110253	338
9	110590	110926	111263	111599	111934	112270	112605	112940	113275	3609	335
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4	7105	7429	7753	8076	8399	8722	9045	9368	9690	130012	323
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7	5900	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
8	8657	8932	9206	9481	9755	200029	200303	200577	200850	201124	274
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3	212188	212454	2720	2986	3252	3518	3783	4049	4314	4579	266
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9	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229
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4	350248	350442	350636	350829	351023	351216	351410	351603	351796	1989	193
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4	9216	9401	9587	9772	9958	370143	370328	370513	370698	370883	185
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7	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394	183
8	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216	182
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6	390935	391112	391288	391464	391641	1817	1993	2169	2345	2521	176
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9	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766	174
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2	401401	401573	1745	1917	2089	2261	2433	2605	2777	2949	172
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7	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410	141
8	8551	8692	8833	8974	9114	9255	9396	9537	9677	9818	141
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7	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701	79
8	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493	79
9	9572	9651	9731	9810	9889	9968	740047	740126	740205	740284	79
550	740363	740442	740521	740600	740678	740757	740836	740915	740994	741073	79
1	1152	1230	1309	1388	1467	1546	1624	1703	1782	1860	79
2	1939	2018	2096	2175	2254	2332	2411	2489	2568	2647	79
3	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431	78
4	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215	78
555	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997	78
6	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	78
7	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556	78
8	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334	78
9	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110	78
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1	8963	9040	9118	9195	9272	9350	9427	9504	9582	9659	77
2	9736	9814	9891	9968	750045	750123	750200	750277	750354	750431	77
3	750508	750586	750663	750740	0817	0894	0971	1048	1125	1202	77
4	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972	77
565	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740	77
6	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506	77
7	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272	77
8	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036	76
9	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799	76
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1	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320	76
2	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079	76
3	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836	76
4	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592	76
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6	760422	760498	760573	760649	760724	0799	0875	0950	1025	1101	75
7	1176	1251	1326	1402	1477	1552	1627	1702	1778	1853	75
8	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604	75
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2	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594	75
3	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338	74
4	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082	74
585	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823	74
6	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564	74
7	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303	74
8	9377	9451	9525	9599	9673	9746	9820	9894	9968	770042	74
9	770115	770189	770263	770336	770410	770484	770557	770631	770705	0778	74
590	770852	770926	770999	771073	771146	771220	771293	771367	771440	771514	74
1	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248	73
2	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981	73
3	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713	73
4	3786	3860	3933	4006	4079	4152	4225	4208	4371	4444	73
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6	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	73
7	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629	73
8	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354	73
9	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079	72
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2	9596	9669	9741	9813	9885	9957	780029	780101	780173	780245	72
3	780317	780389	780461	780533	780605	780677	0749	0821	0893	0965	72
4	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684	72
605	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401	72
6	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117	72
7	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832	71
8	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546	71
9	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259	71
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1	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680	71
2	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390	71
3	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098	71
4	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804	71
615	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510	71
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7	790285	790356	790426	790496	790567	790637	0707	0778	0848	0918	70
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9	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322	70
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3	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115	70
4	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811	70
625	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505	69
6	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198	69
7	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	69
8	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582	69
9	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272	69
630	799341	799409	799478	799547	799616	799685	799754	799823	799892	799961	69
1	800029	800098	800167	800236	800305	800373	800442	800511	800580	800648	69
2	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335	69
3	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	69
4	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	68
635	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389	68
6	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071	68
7	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	68
8	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433	68
9	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112	68
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2	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143	68
3	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818	67
4	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492	67
645	9560	9627	9694	9762	9829	9896	9964	810031	810098	810165	67
6	810233	810300	810367	810434	810501	810569	810636	0703	0770	0837	67
7	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508	67
8	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178	67
9	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847	67
650	812913	812980	813047	813114	813181	813247	813314	813381	813448	813514	67
1	3581	3648	3714	3781	3848	3914	3981	4048	4114	4181	67
2	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847	67
3	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	66
4	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	66
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6	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499	66
7	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160	66
8	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820	66
9	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478	66
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3	1514	1579	1645	1710	1775	1841	1906	1972	2037	2103	65
4	2168	2233	2299	2364	2430	2495	2560	2626	2691	2756	65
665	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409	65
6	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061	65
7	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	65
8	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361	65
9	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010	65
670	826075	826140	826204	826269	826334	826399	826464	826528	826593	826658	65
1	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305	65
2	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951	65
3	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595	64
4	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239	64
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6	9947	830011	830075	830139	830204	830268	830332	830396	830460	830525	64
7	830589	0653	0717	0781	0845	0909	0973	1037	1102	1166	64
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9	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445	64
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4	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627	63
685	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261	63
6	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894	63
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8	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156	63
9	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	63
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6	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170	62
7	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793	62
8	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415	62
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1	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275	62
2	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894	62
3	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511	62
4	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128	62
705	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743	62
6	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	61
7	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972	61
8	850033	850095	850156	850217	850279	850340	850401	850462	850524	850585	61
9	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197	61
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1	1870	1931	1992	2053	2114	2175	2236	2297	2358	2419	61
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4	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245	61
715	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852	61
6	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459	61
7	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064	61
8	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668	60
9	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272	60
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1	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477	60
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6	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475	60
7	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072	60
8	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668	60
9	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263	60
730	863323	863382	863442	863501	863561	863620	863680	863739	863799	863858	59
1	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452	59
2	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045	59
3	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637	59
4	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228	59
735	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819	59
6	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409	59
7	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998	59
8	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586	59
9	8644	8703	8762	8821	8879	8938	8997	9056	9114	9173	59
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1	9818	9877	9935	9994	870053	870111	870170	870228	870287	870345	59
2	870404	870462	870521	870579	0638	0696	0755	0813	0872	0930	58
3	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515	58
4	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098	58
745	2156	2215	2273	2331	2389	2448	2506	2564	2622	2681	58
6	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262	58
7	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844	58
8	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424	58
9	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003	58
750	875061	875119	875177	875235	875293	875351	875409	875466	875524	875582	58
1	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160	58
2	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737	58
3	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314	58
4	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889	58
755	7947	8004	8062	8119	8177	8234	8292	8349	8407	8464	57
6	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039	57
7	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612	57
8	9669	9726	9784	9841	9898	9956	880013	880070	880127	880185	57
9	880242	880299	880356	880413	880471	880528	0585	0642	0699	0756	57
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1	1385	1442	1499	1556	1613	1670	1727	1784	1841	1898	57
2	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468	57
3	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037	57
4	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605	57
765	3661	3718	3775	3832	3888	3945	4002	4059	4115	4172	57
6	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739	57
7	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305	57
8	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870	57
9	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434	56
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3	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685	56
4	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246	56
775	9302	9358	9414	9470	9526	9582	9638	9694	9750	9806	56
6	9862	9918	9974	890030	890086	890141	890197	890253	890309	890365	56
7	890421	890477	890533	0589	0645	0700	0756	0812	0868	0924	56
8	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482	56
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4	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814	55
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6	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	55
7	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471	55
8	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	55
9	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	55
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2	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218	55
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9	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036	54
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2	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661	54
3	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202	54
4	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	54
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6	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	54
7	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358	54
8	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	54
9	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	54
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3	910091	910144	910197	910251	910304	910358	910411	910464	910518	0571	53
4	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104	53
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6	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	53
7	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	53
8	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	53
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4	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634	52
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9	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228	52
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7	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345	51
8	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857	51
9	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368	51
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4	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966	50
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9	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469	50
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4	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958	50
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6	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950	50
7	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445	49
8	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939	49
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2	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912	49
3	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403	49
4	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894	49
885	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385	49
6	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875	49
7	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364	49
8	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853	49
9	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341	49
890	949390	949439	949488	949536	949585	949634	949683	949731	949780	949829	49
1	9878	9926	9975	950024	950073	950121	950170	950219	950267	950316	49
2	950365	950414	950462	0511	0560	0608	0657	0706	0754	0803	49
3	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289	49
4	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	49
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6	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744	48
7	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228	48
8	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711	48
9	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194	48
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4	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601	48
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6	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559	48
7	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038	48
8	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516	48
9	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	48
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2	9995	960042	960090	960138	960185	960233	960281	960328	960376	960423	48
3	960471	0518	0566	0613	0661	0709	0756	0804	0851	0899	48
4	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374	48
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6	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	47
7	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795	47
8	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268	47
9	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741	47
920	963788	963835	963882	963929	963977	964024	964071	964118	964165	964212	47
1	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684	47
2	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	47
3	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	47
4	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	47
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6	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	47
7	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	47
8	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	47
9	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	47
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8	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	46
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4	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	46
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7	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	46
8	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	46
9	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	46
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3	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	46
4	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	46
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6	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	45
7	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	45
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7	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	45
8	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	45
9	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	45
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2	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	45
3	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	45
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7	9895	9939	9983	990028	990072	990117	990161	990206	990250	990294	44
8	990339	990383	990428	0472	0516	0561	0605	0650	0694	0738	44
9	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182	44
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6	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	44
7	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	44
8	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	44
9	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	44
990	995635	995679	995723	995767	995811	995854	995898	995942	995986	996030	44
1	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	44
2	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	44
3	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	44
4	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	44
995	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216	44
6	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	44
7	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	44
8	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	44
9	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	43
N.	0	1	2	3	4	5	6	7	8	9	D.

Table 2

COMPOUND AMOUNT OF 1

$$s = (1 + i)^n$$

n	1/8 %	1/4 %	7/24 %	1/3 %	3/8 %
1	1.0012 5000	1.0025 0000	1.0029 1667	1.0033 3333	1.0037 5000
2	1.0025 0156	1.0050 0625	1.0058 4184	1.0066 7778	1 0075 1406
3	1.0037 5469	1 0075 1877	1.0087 7555	1.0100 3337	1.0112 9224
4	1.0050 0938	1.0100 3756	1.0117 1781	1 0134 0015	1.0150 8459
5	1.0062 6564	1.0125 6266	1.0146 6865	1.0167 7815	1 0188 9115
6	1.0075 2348	1.0150 9406	1.0176 2810	1.0201 6740	1 0227 1200
7	1.0087 8288	1.0176 3180	1.0205 9618	1.0235 6797	1.0265 4717
8	1.0100 4386	1.0201 7588	1 0235 7292	1.0269 7986	1 0303 9672
9	1.0113 0641	1.0227 2632	1.0265 5834	1.0304 0313	1.0342 6070
10	1.0125 7055	1.0252 8313	1.0295 5247	1.0338 3780	1.0381 3918
11	1.0138 3626	1.0278 4634	1.0325 5533	1.0372 8393	1 0420 3220
12	1.0151 0356	1.0304 1596	1 0355 6695	1.0407 4154	1 0459 3983
13	1.0163 7244	1.0329 9200	1.0385 8736	1.0442 1068	1.0498 6210
14	1.0176 4290	1.0355 7448	1.0416 1657	1.0476 9138	1 0537 9908
15	1 0189 1495	1.0381 6341	1.0446 5462	1.0511 8369	1.0577 5083
16	1.0201 8860	1 0407 5882	1.0477 0153	1 0546 8763	1 0617 1739
17	1.0214 6383	1.0433 6072	1.0507 5732	1.0582 0326	1 0656 9883
18	1 0227 4066	1.0459 6912	1.0538 2203	1.0617 3060	1 0696 9521
19	1.0240 1909	1.0485 8404	1.0568 9568	1.0652 6971	1 0737 0656
20	1.0252 9911	1.0512 0550	1.0599 7829	1.0688 2060	1 0777 3296
21	1.0265 8074	1 0538 3352	1.0630 6990	1 0723 8334	1 0817 7446
22	1.0278 6396	1.0564 6810	1 0661 7052	1 0759 5795	1 0858 3111
23	1.0291 4879	1 0591 0927	1.0692 8018	1 0795 4448	1 0899 0298
24	1 0304 3523	1 0617 5704	1.0723 9891	1 0831 4296	1 0939 9012
25	1.0317 2327	1.0644 1144	1 0755 2674	1.0867 5344	1.0980 9258
26	1.0330 1293	1.0670 7247	1.0786 6370	1 0903 7595	1 1022 1043
27	1.0343 0419	1 0697 4015	1 0818 0980	1 0940 1053	1 1063 4372
28	1 0355 9707	1 0724 1450	1 0849 6508	1 0976 5724	1 1104 9251
29	1.0368 9157	1 0750 9553	1 0881 2956	1 1013 1609	1 1146 5685
30	1.0381 8768	1 0777 8327	1.0913 0327	1 1049 8715	1 1188 3682
31	1.0394 8542	1.0804 7773	1.0944 8624	1.1086 7044	1.1230 3245
32	1 0407 8478	1.0831 7892	1.0976 7849	1.1123 6601	1 1272 4383
33	1.0420 8576	1 0858 8687	1 1008 8005	1.1160 7389	1 1314 7099
34	1 0433 8836	1.0886 0159	1 1040 9095	1.1197 9414	1 1357 1401
35	1.0446 9260	1.0913 2309	1.1073 1122	1.1235 2679	1.1399 7293
36	1.0459 9847	1 0940 5140	1.1105 4088	1.1272 7187	1 1442 4783
37	1.0473 0596	1.0967 8653	1.1137 7995	1 1310 2945	1.1485 3876
38	1.0486 1510	1.0995 2850	1.1170 2848	1.1347 9955	1.1528 4578
39	1.0499 2586	1.1022 7732	1.1202 8648	1.1385 8221	1.1571 6895
40	1.0512 3827	1.1050 3301	1.1235 5398	1.1423 7748	1.1615 0834
41	1.0525 5232	1.1077 9559	1.1268 3101	1.1461 8541	1.1658 6399
42	1 0538 6801	1.1105 6508	1.1301 1760	1.1500 0603	1.1702 3598
43	1.0551 8535	1.1133 4149	1.1334 1378	1.1538 3938	1.1746 2437
44	1.0565 0433	1.1161 2485	1.1367 1957	1.1576 8551	1.1790 2921
45	1.0578 2496	1.1189 1516	1.1400 3500	1.1615 4446	1.1834 5057
46	1.0591 4724	1.1217 1245	1.1433 6010	1.1654 1628	1.1878 8851
47	1.0604 7117	1.1245 1673	1.1466 9490	1.1693 0100	1.1923 4309
48	1.0617 9676	1.1273 2802	1.1500 3943	1 1731 9867	1 1968 1438
49	1 0631 2401	1.1301 4634	1.1533 9371	1 1771 0933	1 2013 0243
50	1.0644 5291	1.1329 7171	1.1567 5778	1 1810 3303	1.2058 0732

$$s = (1 + i)^n$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{5}{8}\%$	$\frac{2}{3}\%$
1	1.0041 6667	1.0050 0000	1.0058 3333	1.0062 5000	1.0066 6667
2	1.0083 5069	1.0100 2500	1.0117 0069	1.0125 3906	1.0133 7778
3	1.0125 5216	1.0150 7513	1.0176 0228	1.0188 6743	1.0201 3363
4	1.0167 7112	1.0201 5050	1.0235 3830	1.0252 3535	1.0269 3452
5	1.0210 0767	1.0252 5125	1.0295 0894	1.0316 4307	1.0337 8075
6	1.0252 6187	1.0303 7751	1.0355 1440	1.0380 9084	1.0406 7262
7	1.0295 3379	1.0355 2940	1.0415 5490	1.0445 7891	1.0476 1047
8	1.0338 2352	1.0407 0704	1.0476 3064	1.0511 0753	1.0545 9451
9	1.0381 3111	1.0459 1058	1.0537 4182	1.0576 7695	1.0616 2514
10	1.0424 5666	1.0511 4013	1.0598 8865	1.0642 8743	1.0687 0264
11	1.0468 0023	1.0563 9583	1.0660 7133	1.0709 3923	1.0758 2732
12	1.0511 6180	1.0616 7781	1.0722 9008	1.0776 3260	1.0829 9951
13	1.0555 4173	1.0669 8620	1.0785 4511	1.0843 6780	1.0902 1950
14	1.0599 3983	1.0723 2113	1.0848 3662	1.0911 4510	1.0974 8763
15	1.0643 5625	1.0776 8274	1.0911 6483	1.0979 6476	1.1048 0422
16	1.0687 9106	1.0830 7115	1.0975 2996	1.1048 2704	1.1121 6958
17	1.0732 4436	1.0884 8651	1.1039 3222	1.1117 3221	1.1195 8404
18	1.0777 1621	1.0939 2894	1.1103 7182	1.1186 8053	1.1270 4794
19	1.0822 0670	1.0993 9858	1.1168 4899	1.1256 7229	1.1345 6159
20	1.0867 1589	1.1048 9558	1.1233 6395	1.1327 0774	1.1421 2533
21	1.0912 4387	1.1104 2006	1.1299 1690	1.1397 8716	1.1497 3950
22	1.0957 9072	1.1159 7216	1.1365 0808	1.1469 1083	1.1574 0443
23	1.1003 5652	1.1215 5202	1.1431 3771	1.1540 7902	1.1651 2046
24	1.1049 4134	1.1271 5978	1.1498 0602	1.1612 9202	1.1728 8793
25	1.1095 4526	1.1327 9558	1.1565 1322	1.1685 5009	1.1807 0718
26	1.1141 6836	1.1384 5955	1.1632 5955	1.1758 5353	1.1885 7857
27	1.1188 1073	1.1441 5185	1.1700 4523	1.1832 0262	1.1965 0242
28	1.1234 7244	1.1498 7261	1.1768 7049	1.1905 9763	1.2044 7911
29	1.1281 5358	1.1556 2197	1.1837 3557	1.1980 3887	1.2125 0897
30	1.1328 5422	1.1614 0008	1.1906 4069	1.2055 2661	1.2205 9236
31	1.1375 7444	1.1672 0708	1.1975 8610	1.2130 6115	1.2287 2964
32	1.1423 1434	1.1730 4312	1.2045 7202	1.2206 4278	1.2369 2117
33	1.1470 7398	1.1789 0833	1.2115 9869	1.2282 7180	1.2451 6731
34	1.1518 5346	1.1848 0288	1.2186 6634	1.2359 4850	1.2534 6843
35	1.1566 5284	1.1907 2689	1.2257 7523	1.2436 7318	1.2618 2489
36	1.1614 7223	1.1966 8052	1.2329 2559	1.2514 4614	1.2702 3705
37	1.1663 1170	1.2026 6393	1.2401 1765	1.2592 6767	1.2787 0530
38	1.1711 7133	1.2086 7725	1.2473 5167	1.2671 3810	1.2872 3000
39	1.1760 5121	1.2147 2063	1.2546 2789	1.2750 5771	1.2958 1153
40	1.1809 5143	1.2207 9424	1.2619 4655	1.2830 2682	1.3044 5028
41	1.1858 7206	1.2268 9821	1.2693 0791	1.2910 4574	1.3131 4661
42	1.1908 1319	1.2330 3270	1.2767 1220	1.2991 1477	1.3219 0092
43	1.1957 7491	1.2391 9786	1.2841 5969	1.3072 3424	1.3307 1360
44	1.2007 5731	1.2453 9385	1.2916 5062	1.3154 0446	1.3395 8502
45	1.2057 6046	1.2516 2082	1.2991 8525	1.3236 2573	1.3485 1559
46	1.2107 8446	1.2578 7892	1.3067 6383	1.3318 9839	1.3575 0569
47	1.2158 2940	1.2641 6832	1.3143 8662	1.3402 2276	1.3665 5573
48	1.2208 9536	1.2704 8916	1.3220 5388	1.3485 9915	1.3756 6610
49	1.2259 8242	1.2768 4161	1.3297 6586	1.3570 2790	1.3848 3721
50	1.2310 9068	1.2832 2581	1.3375 2283	1.3655 0932	1.3940 6946

$$s = (1 + i)^n$$

n	¾%	1%	1⅛%	1¼%	1⅜%
1	1.0075	1.01	1.0112 5	1.0125	1.0137 5
2	1.0150 5625	1.0201	1.0226 2656	1.0251 5625	1.0276 8906
3	1.0226 6917	1.0303 01	1.0341 3111	1.0379 7070	1.0418 1979
4	1.0303 3919	1.0406 0401	1.0457 6509	1.0509 4531	1.0561 4481
5	1.0380 6673	1.0510 1005	1.0575 2994	1.0640 8215	1.0706 6680
6	1.0458 5224	1.0615 2015	1.0694 2716	1.0773 8318	1.0853 8847
7	1.0536 9613	1.0721 3535	1.0814 5821	1.0908 5047	1.1003 1256
8	1.0615 9885	1.0828 5671	1.0936 2462	1.1044 8610	1.1154 4186
9	1.0695 6084	1.0936 8527	1.1059 2789	1.1182 9218	1.1307 7918
10	1.0775 8255	1.1046 2213	1.1183 6958	1.1322 7083	1.1463 2740
11	1.0856 6441	1.1156 6835	1.1309 5124	1.1464 2422	1.1620 8940
12	1.0938 0690	1.1268 2503	1.1436 7414	1.1607 5452	1.1780 6813
13	1.1020 1045	1.1380 9328	1.1565 4078	1.1752 6395	1.1942 6656
14	1.1102 7553	1.1494 7421	1.1695 5186	1.1899 5475	1.2106 8773
15	1.1186 0259	1.1609 6896	1.1827 0932	1.2048 2918	1.2273 3469
16	1.1269 9211	1.1725 7864	1.1960 1480	1.2198 8955	1.2442 1054
17	1.1354 4455	1.1843 0443	1.2094 6997	1.2351 3817	1.2613 1843
18	1.1439 6039	1.1961 4748	1.2230 7650	1.2505 7739	1.2786 6156
19	1.1525 4009	1.2081 0895	1.2368 3611	1.2662 0961	1.2962 4316
20	1.1611 8414	1.2201 9004	1.2507 5052	1.2820 3723	1.3140 6650
21	1.1698 9302	1.2323 9194	1.2648 2146	1.2980 6270	1.3321 3492
22	1.1786 6722	1.2447 1586	1.2790 5071	1.3142 8848	1.3504 5177
23	1.1875 0723	1.2571 6302	1.2934 4003	1.3307 1709	1.3690 2048
24	1.1964 1353	1.2697 3465	1.3079 9123	1.3473 5105	1.3878 4151
25	1.2053 8663	1.2824 3200	1.3227 0613	1.3641 9294	1.4069 2738
26	1.2144 2703	1.2952 5631	1.3375 8657	1.3812 4535	1.4262 7263
27	1.2235 3523	1.3082 0888	1.3526 3442	1.3985 1092	1.4458 8388
28	1.2327 1175	1.3212 9097	1.3678 5156	1.4159 9230	1.4657 6478
29	1.2419 5709	1.3345 0388	1.3832 3989	1.4336 9221	1.4859 1905
30	1.2512 7176	1.3478 4892	1.3988 0134	1.4516 1336	1.5063 5043
31	1.2606 5630	1.3613 2740	1.4145 3785	1.4697 5853	1.5270 6275
32	1.2701 1122	1.3749 4068	1.4304 5140	1.4881 3051	1.5480 5986
33	1.2796 3706	1.3886 9009	1.4465 4398	1.5067 3214	1.5693 4569
34	1.2892 3434	1.4025 7699	1.4628 1760	1.5255 6629	1.5909 2419
35	1.2989 0359	1.4166 0276	1.4792 7430	1.5446 3587	1.6127 9940
36	1.3086 4537	1.4307 6878	1.4959 1613	1.5639 4382	1.6349 7539
37	1.3184 6021	1.4450 7647	1.5127 4519	1.5834 9312	1.6574 5630
38	1.3283 4866	1.4595 2724	1.5297 6357	1.6032 8678	1.6802 4633
39	1.3383 1128	1.4741 2251	1.5469 7341	1.6233 2787	1.7033 4971
40	1.3483 4861	1.4888 6373	1.5643 7687	1.6436 1946	1.7267 7077
41	1.3584 6123	1.5037 5237	1.5819 7611	1.6641 6471	1.7505 1387
42	1.3686 4969	1.5187 8989	1.5997 7334	1.6849 6677	1.7745 8343
43	1.3789 1456	1.5339 7779	1.6177 7079	1.7060 2885	1.7989 8396
44	1.3892 5642	1.5493 1757	1.6359 7071	1.7273 5421	1.8237 1999
45	1.3996 7584	1.5648 1075	1.6543 7538	1.7489 4614	1.8487 9614
46	1.4101 7341	1.5804 5885	1.6729 8710	1.7708 0797	1.8742 1708
47	1.4207 4971	1.5962 6344	1.6918 0821	1.7929 4306	1.8999 8757
48	1.4314 0533	1.6122 2608	1.7108 4105	1.8153 5485	1.9261 1240
49	1.4421 4087	1.6283 4834	1.7300 8801	1.8380 4679	1.9525 9644
50	1.4529 5693	1.6446 3182	1.7495 5150	1.8610 2237	1.9794 4464

$$s = (1 + i)^n$$

n	$1\frac{1}{2}\%$	$1\frac{5}{8}\%$	$1\frac{3}{4}\%$	$1\frac{7}{8}\%$	2%
1	1.015	1.0162 5	1.0175	1.0187 5	1.02
2	1.0302 25	1.0327 6406	1.0353 0625	1.0378 5156	1.0404
3	1.0456 7838	1.0495 4648	1.0534 2411	1.0573 1128	1.0612 08
4	1.0613 6355	1.0666 0161	1.0718 5903	1.0771 3587	1.0824 3216
5	1.0772 8400	1.0839 3388	1.0906 1656	1.0973 3216	1.1040 8080
6	1.0934 4326	1.1015 4781	1.1097 0235	1.1179 0714	1.1261 6242
7	1.1098 4491	1.1194 4796	1.1291 2215	1.1388 6790	1.1486 8567
8	1.1264 9259	1.1376 3809	1.1488 8178	1.1602 2167	1.1716 5938
9	1.1433 8998	1.1561 2563	1.1689 8721	1.1819 7583	1.1950 9257
10	1.1605 4083	1.1749 1267	1.1894 4449	1.2041 3788	1.2189 9442
11	1.1779 4894	1.1940 0500	1.2102 5977	1.2267 1546	1.2433 7431
12	1.1956 1817	1.2134 0758	1.2314 3931	1.2497 1638	1.2682 4179
13	1.2135 5244	1.2331 2545	1.2529 8950	1.2731 4856	1.2936 0663
14	1.2317 5573	1.2531 6374	1.2749 1682	1.2970 2009	1.3194 7876
15	1.2502 3207	1.2735 2765	1.2972 2786	1.3213 3922	1.3458 6834
16	1.2689 8555	1.2942 2248	1.3199 2935	1.3461 1433	1.3727 8571
17	1.2880 2033	1.3152 5359	1.3430 2811	1.3713 5308	1.4002 4142
18	1.3073 4064	1.3366 2646	1.3665 3111	1.3970 6686	1.4282 4625
19	1.3269 5075	1.3583 4664	1.3904 4540	1.4232 6187	1.4568 1117
20	1.3468 5501	1.3804 4977	1.4147 7820	1.4499 4803	1.4859 4740
21	1.3670 5783	1.4028 5160	1.4395 3681	1.4771 3455	1.5156 6634
22	1.3875 6370	1.4256 4793	1.4647 2871	1.5048 3082	1.5459 7967
23	1.4083 7715	1.4488 1471	1.4903 6146	1.5330 4640	1.5768 9926
24	1.4295 0281	1.4723 5795	1.5164 4279	1.5617 9102	1.6084 3725
25	1.4509 4535	1.4962 8377	1.5429 8054	1.5910 7460	1.6406 0599
26	1.4727 0953	1.5205 9838	1.5699 8269	1.6209 0725	1.6734 1811
27	1.4948 0018	1.5453 0810	1.5974 5739	1.6512 9926	1.7068 8648
28	1.5172 2218	1.5704 1936	1.6254 1290	1.6822 6112	1.7410 2421
29	1.5399 8051	1.5959 3868	1.6538 5762	1.7138 0352	1.7758 4469
30	1.5630 8022	1.6218 7268	1.6828 0013	1.7459 3734	1.8113 6158
31	1.5865 2642	1.6482 2811	1.7122 4913	1.7786 7366	1.8475 8882
32	1.6103 2432	1.6750 1182	1.7422 1349	1.8120 2379	1.8845 4059
33	1.6344 7918	1.7022 3076	1.7727 0223	1.8459 9924	1.9222 3140
34	1.6589 9637	1.7298 9201	1.8037 2452	1.8806 1172	1.9606 7603
35	1.6838 8132	1.7580 0275	1.8352 8970	1.9158 7319	1.9998 8955
36	1.7091 3954	1.7865 7030	1.8674 0727	1.9517 9582	2.0398 8734
37	1.7347 7663	1.8156 0207	1.9000 8689	1.9883 9199	2.0806 8509
38	1.7607 9828	1.8451 0560	1.9333 3841	2.0256 7434	2.1222 9879
39	1.7872 1025	1.8750 8857	1.9671 7184	2.0636 5573	2.1647 4477
40	1.8140 1841	1.9055 5875	2.0015 9734	2.1023 4928	2.2080 3966
41	1.8412 2868	1.9365 2408	2.0366 2530	2.1417 6833	2.2522 0046
42	1.8688 4712	1.9679 9260	2.0722 6624	2.1819 2618	2.2972 4447
43	1.8968 7982	1.9999 7248	2.1085 3090	2.2228 3760	2.3431 8936
44	1.9253 3302	2.0324 7203	2.1454 3019	2.2645 1581	2.3900 5314
45	1.9542 1301	2.0654 9970	2.1829 7522	2.3069 7548	2.4378 5421
46	1.9835 2621	2.0990 6407	2.2211 7728	2.3502 3127	2.4866 1129
47	2.0132 7910	2.1331 7387	2.2600 4789	2.3942 9811	2.5363 4351
48	2.0434 7829	2.1678 3794	2.2995 9872	2.4391 9120	2.5870 7039
49	2.0741 3046	2.2030 6531	2.3398 4170	2.4849 2603	2.6388 1179
50	2.1052 4242	2.2388 6512	2.3807 8893	2.5315 1839	2.6915 8803

$$s = (1 + i)^n$$

n	2⅛%	2¼%	2⅜%	2½%	2¾%
1	1.0212 5	1.0225	1.0237 5	1.025	1.0275
2	1.0429 5156	1.0455 0625	1.0480 6406	1.0506 25	1.0557 5625
3	1.0651 1428	1.0690 3014	1.0729 5558	1.0768 9063	1.0847 8955
4	1.0877 4796	1.0930 8332	1.0984 3828	1.1038 1289	1.1146 2126
5	1.1108 6261	1.1176 7769	1.1245 2619	1.1314 0821	1.1452 7334
6	1.1344 6844	1.1428 2544	1.1512 3369	1.1596 9342	1.1767 6836
7	1.1585 7589	1.1685 3901	1.1785 7519	1.1886 8575	1.2091 2949
8	1.1831 9563	1.1948 3114	1.2065 6665	1.2184 0290	1.2423 8055
9	1.2083 3854	1.2217 1484	1.2352 2261	1.2488 6297	1.2765 4602
10	1.2340 1573	1.2492 0343	1.2645 5915	1.2800 8454	1.3116 5103
11	1.2602 3856	1.2773 1050	1.2945 9243	1.3120 8666	1.3477 2141
12	1.2870 1863	1.3060 4999	1.3253 3900	1.3448 8882	1.3847 8378
13	1.3143 6778	1.3354 3611	1.3568 1580	1.3785 1104	1.4228 6533
14	1.3422 9809	1.3654 8343	1.3890 4017	1.4129 7382	1.4619 9413
15	1.3708 2193	1.3962 0680	1.4220 2988	1.4482 9817	1.5021 9896
16	1.3999 5189	1.4276 2146	1.4558 0309	1.4845 0562	1.5435 0944
17	1.4297 0087	1.4597 4294	1.4903 7841	1.5216 1826	1.5859 5595
18	1.4600 8202	1.4925 8716	1.5257 7490	1.5596 5872	1.6295 6973
19	1.4911 0876	1.5261 7037	1.5620 1205	1.5986 5019	1.6743 8290
20	1.5227 9482	1.5605 0920	1.5991 0984	1.6386 1644	1.7204 2843
21	1.5551 5421	1.5956 2066	1.6370 8870	1.6795 8185	1.7677 4021
22	1.5882 0124	1.6315 2212	1.6759 6955	1.7215 7140	1.8163 5307
23	1.6219 5051	1.6682 3137	1.7157 7383	1.7646 1068	1.8663 0278
24	1.6564 1696	1.7057 6658	1.7565 2346	1.8087 2595	1.9176 2610
25	1.6916 1582	1.7441 4632	1.7982 4089	1.8539 4410	1.9703 6082
26	1.7275 6266	1.7833 8962	1.8409 4911	1.9002 9270	2.0245 4575
27	1.7642 7336	1.8235 1588	1.8846 7165	1.9478 0002	2.0802 2075
28	1.8017 6417	1.8645 4499	1.9294 3261	1.9964 9502	2.1374 2682
29	1.8400 5166	1.9064 9725	1.9752 5663	2.0464 0739	2.1962 0606
30	1.8791 5276	1.9493 9344	2.0221 6898	2.0975 6758	2.2566 0173
31	1.9190 8476	1.9932 5479	2.0701 9549	2.1500 0677	2.3186 5828
32	1.9598 6531	2.0381 0303	2.1193 6263	2.2037 5694	2.3824 2138
33	2.0015 1245	2.0839 6034	2.1696 9749	2.2588 5086	2.4479 3797
34	2.0440 4458	2.1308 4945	2.2212 2781	2.3153 2213	2.5152 5626
35	2.0874 8053	2.1787 9356	2.2739 8197	2.3732 0519	2.5844 2581
36	2.1318 3949	2.2278 1642	2.3279 8904	2.4325 3532	2.6554 9752
37	2.1771 4108	2.2779 4229	2.3832 7878	2.4933 4870	2.7285 2370
38	2.2234 0533	2.3291 9599	2.4398 8165	2.5556 8242	2.8035 5810
39	2.2706 5269	2.3816 0290	2.4978 2884	2.6195 7448	2.8806 5595
40	2.3189 0406	2.4351 8897	2.5571 5228	2.6850 6384	2.9598 7399
41	2.3681 8077	2.4899 8072	2.6178 8464	2.7521 9043	3.0412 7052
42	2.4185 0462	2.5460 0528	2.6800 5940	2.8209 9520	3.1249 0546
43	2.4698 9784	2.6032 9040	2.7437 1081	2.8915 2008	3.2108 4036
44	2.5223 8317	2.6618 6444	2.8088 7395	2.9638 0808	3.2991 3847
45	2.5759 8381	2.7217 5639	2.8755 8470	3.0379 0328	3.3898 6478
46	2.6307 2347	2.7829 9590	2.9438 7984	3.1138 5086	3.4830 8606
47	2.6866 2634	2.8456 1331	3.0137 9699	3.1916 9713	3.5788 7093
48	2.7437 1715	2.9096 3961	3.0853 7466	3.2714 8956	3.6772 8988
49	2.8020 2114	2.9751 0650	3.1586 5231	3.3532 7680	3.7784 1535
50	2.8615 6409	3.0420 4640	3.2336 7030	3.4371 0872	3.8823 2177

$$s = (1 + i)^n$$

n	3%	3¼%	3½%	3¾%	4%
1	1.03	1.0325	1.035	1.0375	1.04
2	1.0609	1.0660 5625	1.0712 25	1.0764 0625	1.0816
3	1.0927 27	1.1007 0308	1.1087 1788	1.1167 7148	1.1248 64
4	1.1255 0881	1.1364 7593	1.1475 2300	1.1586 5042	1.1698 5856
5	1.1592 7407	1.1734 1140	1.1876 8631	1.2020 9981	1.2166 5290
6	1.1940 5230	1.2115 4727	1.2292 5533	1.2471 7855	1.2653 1902
7	1.2298 7387	1.2509 2255	1.2722 7926	1.2939 4774	1.3159 3178
8	1.2667 7008	1.2915 7754	1.3168 0904	1.3424 7078	1.3685 6905
9	1.3047 7318	1.3335 5381	1.3628 9735	1.3928 1344	1.4233 1181
10	1.3439 1638	1.3768 9430	1.4105 9876	1.4450 4394	1.4802 4428
11	1.3842 3387	1.4216 4337	1.4599 6972	1.4992 3309	1.5394 5406
12	1.4257 6089	1.4678 4678	1.5110 6866	1.5554 5433	1.6010 3222
13	1.4685 3371	1.5155 5180	1.5639 5606	1.6137 8387	1.6650 7351
14	1.5125 8972	1.5648 0723	1.6186 9152	1.6743 0076	1.7316 7645
15	1.5579 6742	1.6156 6347	1.6753 4883	1.7370 8704	1.8009 4351
16	1.6047 0644	1.6681 7253	1.7339 8604	1.8022 2781	1.8729 8125
17	1.6528 4763	1.7223 8814	1.7946 7555	1.8698 1135	1.9479 0050
18	1.7024 3306	1.7783 6575	1.8574 8920	1.9399 2927	2.0258 1652
19	1.7535 0605	1.8361 6264	1.9225 0132	2.0126 7662	2.1068 4918
20	1.8061 1123	1.8958 3792	1.9897 8886	2.0881 5200	2.1911 2314
21	1.8602 9457	1.9574 5266	2.0594 3147	2.1664 5770	2.2787 6807
22	1.9161 0341	2.0210 6987	2.1315 1158	2.2476 9986	2.3699 1879
23	1.9735 8651	2.0867 5464	2.2061 1448	2.3319 8860	2.4647 1551
24	2.0327 9411	2.1545 7416	2.2833 2849	2.4194 3818	2.5633 0416
25	2.0937 7793	2.2245 9782	2.3632 4498	2.5101 6711	2.6658 3633
26	2.1565 9127	2.2968 9725	2.4459 5856	2.6042 9838	2.7724 6978
27	2.2212 8901	2.3715 4611	2.5315 6711	2.7019 5956	2.8833 6858
28	2.2879 2768	2.4486 2167	2.6201 7196	2.8032 8305	2.9987 0332
29	2.3565 6551	2.5282 0188	2.7118 7798	2.9084 0616	3.1186 5145
30	2.4272 6247	2.6103 6844	2.8067 9370	3.0174 7139	3.2433 9751
31	2.5000 8035	2.6952 0541	2.9050 3148	3.1306 2657	3.3731 3341
32	2.5750 8276	2.7827 9959	3.0067 0759	3.2480 2507	3.5080 5875
33	2.6523 3524	2.8732 4058	3.1119 4235	3.3698 2601	3.6483 8110
34	2.7319 0530	2.9666 2089	3.2208 6033	3.4961 9448	3.7943 1634
35	2.8138 6245	3.0630 3607	3.3335 9045	3.6273 0178	3.9460 8899
36	2.8982 7833	3.1625 8475	3.4502 6611	3.7633 2559	4.1039 3255
37	2.9852 2668	3.2653 6875	3.5710 2543	3.9044 5030	4.2680 8986
38	3.0747 8348	3.3714 9323	3.6960 1132	4.0508 6719	4.4388 1345
39	3.1670 2698	3.4810 6676	3.8253 7171	4.2027 7471	4.6163 6599
40	3.2620 3779	3.5942 0143	3.9592 5972	4.3603 7876	4.8010 2063
41	3.3598 9893	3.7110 1298	4.0978 3381	4.5238 9296	4.9930 6145
42	3.4606 9589	3.8316 2090	4.2412 5799	4.6935 3895	5.1927 8391
43	3.5645 1677	3.9561 4858	4.3897 0202	4.8695 4666	5.4004 9527
44	3.6714 5227	4.0847 2341	4.5433 4160	5.0521 5466	5.6165 1508
45	3.7815 9584	4.2174 7692	4.7023 5855	5.2416 1046	5.8411 7568
46	3.8950 4372	4.3545 4492	4.8669 4110	5.4381 7085	6.0748 2271
47	4.0118 9503	4.4960 6763	5.0372 8404	5.6421 0226	6.3178 1562
48	4.1322 5188	4.6421 8983	5.2135 8898	5.8536 8109	6.5705 2824
49	4.2562 1944	4.7930 6100	5.3960 6459	6.0731 9413	6.8333 4937
50	4.3839 0602	4.9488 3548	5.5849 2686	6.3009 3891	7.1066 8335

$$s = (1 + i)^n$$

n	4¼%	4½%	4¾%	5%	5½%
1	1.0425	1.045	1.0475	1.05	1.055
2	1.0868 0625	1.0920 25	1.0972 5625	1.1025	1.1130 25
3	1.1329 9552	1.1411 6613	1.1493 7592	1.1576 25	1.1742 4138
4	1.1811 4783	1.1925 1860	1.2039 7128	1.2155 0625	1.2388 2465
5	1.2313 4661	1.2461 8194	1.2611 5991	1.2762 8156	1.3069 6001
6	1.2836 7884	1.3022 6012	1.3210 6501	1.3400 9564	1.3788 4281
7	1.3382 3519	1.3608 6183	1.3838 1560	1.4071 0042	1.4546 7916
8	1.3951 1018	1.4221 0061	1.4495 4684	1.4774 5544	1.5346 8651
9	1.4544 0237	1.4860 9514	1.5184 0031	1.5513 2822	1.6190 9427
10	1.5162 1447	1.5529 6942	1.5905 2433	1.6288 9463	1.7081 4446
11	1.5806 5358	1.6228 5305	1.6660 7423	1.7103 3936	1.8020 9240
12	1.6478 3136	1.6958 8143	1.7452 1276	1.7958 5633	1.9012 0749
13	1.7178 6419	1.7721 9610	1.8281 1037	1.8856 4914	2.0057 7390
14	1.7908 7342	1.8519 4492	1.9149 4561	1.9799 3160	2.1160 9146
15	1.8669 8554	1.9352 8244	2.0059 0552	2.0789 2818	2.2324 7649
16	1.9463 3243	2.0223 7015	2.1011 8604	2.1828 7459	2.3552 6270
17	2.0290 5156	2.1133 7681	2.2009 9237	2.2920 1832	2.4848 0215
18	2.1152 8625	2.2084 7877	2.3055 3951	2.4066 1923	2.6214 6627
19	2.2051 8591	2.3078 6031	2.4150 5264	2.5269 5020	2.7656 4691
20	2.2989 0631	2.4117 1402	2.5297 6764	2.6532 9771	2.9177 5749
21	2.3966 0983	2.5202 4116	2.6499 3160	2.7859 6259	3.0782 3415
22	2.4984 6575	2.6336 5201	2.7758 0335	2.9252 6072	3.2475 3703
23	2.6046 5054	2.7521 6635	2.9076 5401	3.0715 2376	3.4261 5157
24	2.7153 4819	2.8760 1383	3.0457 6758	3.2250 9994	3.6145 8990
25	2.8307 5049	3.0054 3446	3.1904 4154	3.3863 5494	3.8133 9235
26	2.9510 5739	3.1406 7901	3.3419 8751	3.5556 7269	4.0231 2893
27	3.0764 7732	3.2820 0956	3.5007 3192	3.7334 5632	4.2444 0102
28	3.2072 2761	3.4296 9998	3.6670 1668	3.9201 2914	4.4778 4307
29	3.3435 3478	3.5840 3649	3.8411 9998	4.1161 3560	4.7241 2444
30	3.4856 3501	3.7453 1813	4.0236 5698	4.3219 4238	4.9839 5129
31	3.6337 7450	3.9138 5745	4.2147 8068	4.5380 3949	5.2580 6861
32	3.7882 0992	4.0899 8104	4.4149 8276	4.7649 4147	5.5472 6238
33	3.9492 0884	4.2740 3018	4.6246 9445	5.0031 8854	5.8523 6181
34	4.1170 5021	4.4663 6154	4.8443 6743	5.2533 4797	6.1742 4171
35	4.2920 2485	4.6673 4781	5.0744 7488	5.5160 1537	6.5138 2501
36	4.4744 3590	4.8773 7846	5.3155 1244	5.7918 1614	6.8720 8538
37	4.6645 9943	5.0968 6049	5.5679 9928	6.0814 0694	7.2500 5008
38	4.8628 4491	5.3262 1921	5.8324 7925	6.3854 7729	7.6488 0283
39	5.0695 1581	5.5658 9908	6.1095 2201	6.7047 5115	8.0694 8699
40	5.2849 7024	5.8163 6454	6.3997 2431	7.0399 8871	8.5133 0877
41	5.5095 8147	6.0781 0094	6.7037 1121	7.3919 8815	8.9815 4076
42	5.7437 3868	6.3516 1548	7.0221 3750	7.7615 8756	9.4755 2550
43	5.9878 4758	6.6374 3818	7.3556 8903	8.1496 6693	9.9966 7940
44	6.2423 3110	6.9361 2290	7.7050 8426	8.5571 5028	10.5464 9677
45	6.5076 3017	7.2482 4813	8.0710 7576	8.9850 0779	11.1265 5409
46	6.7842 0445	7.5744 1961	8.4544 5186	9.4342 5818	11.7385 1456
47	7.0725 3314	7.9152 6849	8.8560 3832	9.9059 7109	12.3841 3287
48	7.3731 1580	8.2714 5557	9.2767 0014	10.4012 6965	13.0652 6017
49	7.6864 7322	8.6436 7107	9.7173 4340	10.9213 3313	13.7838 4948
50	8.0131 4834	9.0326 3627	10.1789 1721	11.4673 9979	14.5419 6120

$$s = (1 + i)^n$$

n	6%	6½%	7%	8%	9%
1	1.06	1.065	1.07	1.08	1.09
2	1.1236	1.1342 25	1.1449	1.1664	1.1881
3	1.1910 16	1.2079 4963	1.2250 43	1.2597 12	1.2950 29
4	1.2624 7696	1.2864 6635	1.3107 9601	1.3604 8896	1.4115 8161
5	1.3382 2558	1.3700 8666	1.4025 5173	1.4693 2808	1.5386 2395
6	1.4185 1911	1.4591 4230	1.5007 3035	1.5868 7432	1.6771 0011
7	1.5036 3026	1.5539 8655	1.6057 8148	1.7138 2427	1.8280 3912
8	1.5938 4807	1.6549 9567	1.7181 8618	1.8509 3021	1.9925 6264
9	1.6894 7896	1.7625 7039	1.8384 5921	1.9990 0463	2.1718 9328
10	1.7908 4770	1.8771 3747	1.9671 5136	2.1589 2500	2.3673 6367
11	1.8982 9856	1.9991 5140	2.1048 5195	2.3316 3900	2.5804 2641
12	2.0121 9647	2.1290 9624	2.2521 9159	2.5181 7012	2.8126 6478
13	2.1329 2826	2.2674 8750	2.4098 4500	2.7196 2373	3.0658 0461
14	2.2609 0396	2.4148 7418	2.5785 3415	2.9371 9362	3.3417 2703
15	2.3965 5819	2.5718 4101	2.7590 3154	3.1721 6911	3.6424 8246
16	2.5403 5168	2.7390 1067	2.9521 6375	3.4259 4264	3.9703 0588
17	2.6927 7279	2.9170 4637	3.1588 1521	3.7000 1805	4.3276 3341
18	2.8543 3915	3.1066 5438	3.3799 3228	3.9960 1950	4.7171 2042
19	3.0255 9950	3.3085 8691	3.6165 2751	4.3157 0106	5.1416 6125
20	3.2071 3547	3.5236 4506	3.8696 8446	4.6609 5714	5.6044 1077
21	3.3995 6360	3.7526 8199	4.1405 6237	5.0338 3372	6.1088 0774
22	3.6035 3742	3.9966 0632	4.4304 0174	5.4365 4041	6.6586 0043
23	3.8197 4966	4.2563 8573	4.7405 2986	5.8714 6365	7.2578 7447
24	4.0489 3464	4.5330 5081	5.0723 6695	6.3411 8074	7.9110 8317
25	4.2918 7072	4.8276 9911	5.4274 3264	6.8484 7520	8.6230 8066
26	4.5493 8296	5.1414 9955	5.8073 5292	7.3963 5321	9.3991 5792
27	4.8223 4594	5.4756 9702	6.2138 6763	7.9880 6147	10.2450 8213
28	5.1116 8670	5.8316 1733	6.6488 3836	8.6271 0639	11.1671 3952
29	5.4183 8790	6.2106 7245	7.1142 5705	9.3172 7490	12.1721 8208
30	5.7434 9117	6.6143 6616	7.6122 5504	10.0626 5689	13.2676 7847
31	6.0881 0064	7.0442 9996	8.1451 1290	10.8676 6944	14.4617 6953
32	6.4533 8668	7.5021 7946	8.7152 7080	11.7370 8300	15.7633 2879
33	6.8405 8988	7.9898 2113	9.3253 3975	12.6760 4964	17.1820 2838
34	7.2510 2528	8.5091 5950	9.9781 1354	13.6901 3361	18.7284 1093
35	7.6860 8679	9.0622 5487	10.6765 8148	14.7853 4429	20.4139 6792
36	8.1472 5200	9.6513 0143	11.4239 4219	15.9681 7184	22.2512 2503
37	8.6360 8712	10.2786 3603	12.2236 1814	17.2456 2558	24.2538 3528
38	9.1542 5235	10.9467 4737	13.0792 7141	18.6252 7563	26.4366 8046
39	9.7035 0749	11.6582 8595	13.9948 2041	20.1152 9768	28.8159 8170
40	10.2857 1794	12.4160 7453	14.9744 5784	21.7245 2150	31.4094 2005
41	10.9028 6101	13.2231 1938	16.0226 6989	23.4624 8322	34.2362 6786
42	11.5570 3267	14.0826 2214	17.1442 5678	25.3394 8187	37.3175 3197
43	12.2504 5463	14.9979 9258	18.3443 5475	27.3666 4042	40.6761 0984
44	12.9854 8191	15.9728 6209	19.6284 5959	29.5559 7166	44.3369 5973
45	13.7646 1083	17.0110 9813	21.0024 5176	31.9204 4939	48.3272 8610
46	14.5904 8748	18.1168 1951	22.4726 2338	34.4740 8534	52.6767 4185
47	15.4659 1673	19.2944 1278	24.0457 0702	37.2320 1217	57.4176 4862
48	16.3938 7173	20.5485 4961	25.7289 0651	40.2105 7314	62.5852 3700
49	17.3775 0403	21.8842 0583	27.5299 2997	43.4274 1899	68.2179 0833
50	18.4201 5427	23.3066 7868	29.4570 2506	46.9016 1251	74.3575 2008

Table 3
PRESENT VALUE OF 1

$$v^n = \frac{1}{(1+i)^n}$$

n	1/8%	1/4%	3/8%	1/2%	5/8%
1	0.9987 5156	0.9975 0623	0.9962 6401	0.9950 2488	0.9937 8882
2	0.9975 0468	0.9950 1869	0.9925 4198	0 9900 7450	0 9876 1622
3	0.9962 5936	0 9925 3734	0 9888 3385	0 9851 4876	0 9814 8196
4	0.9950 1559	0.9900 6219	0 9851 3958	0 9802 4752	0.9753 8580
5	0.9937 7337	0.9875 9321	0.9814 5911	0.9753 7067	0.9693 2750
6	0 9925 3270	0.9851 3038	0 9777 9238	0 9705 1808	0.9633 0683
7	0.9912 9359	0 9826 7370	0 9741 3936	0.9656 8963	0 9573 2356
8	0 9900 5602	0.9802 2314	0.9704 9999	0.9608 8520	0 9513 7745
9	0 9888 1999	0.9777 7869	0 9668 7421	0 9561 0468	0 9454 6827
10	0.9875 8551	0 9753 4034	0.9632 6198	0 9513 4794	0 9395 9580
11	0.9863 5257	0 9729 0807	0 9596 6324	0 9466 1489	0 9337 5980
12	0 9851 2117	0.9704 8187	0 9560 7795	0 9419 0534	0 9279 6005
13	0.9838 9130	0 9680 6171	0.9525 0605	0 9372 1924	0 9221 9632
14	0.9826 6297	0 9656 4759	0.9489 4750	0 9325 5646	0.9164 6840
15	0 9814 3618	0.9632 3949	0 9451 0224	0.9279 1688	0.9107 7604
16	0.9802 1092	0 9608 3740	0 9418 7022	0.9233 0037	0 9051 1905
17	0.9789 8718	0.9584 4130	0.9383 5141	0 9187 0684	0 8994 9719
18	0 9777 6498	0.9560 5117	0 9348 4573	0 9141 3616	0.8939 1025
19	0.9765 4430	0.9536 6700	0 9313 5316	0.9095 8822	0.8883 5802
20	0.9753 2514	0.9512 8878	0.9278 7363	0.9050 6290	0.8828 4027
21	0.9741 0750	0.9489 1649	0 9214 0711	0 9005 6010	0.8773 5679
22	0.9728 9139	0 9465 5011	0 9209 5353	0 8960 7971	0.8719 0736
23	0.9716 7679	0 9441 8964	0 9175 1286	0 8916 2160	0 8664 9179
24	0 9704 6371	0.9418 3505	0.9140 8504	0.8871 8567	0 8611 0985
25	0.9692 5215	0.9394 8634	0.9106 7003	0.8827 7181	0.8557 6135
26	0.9680 4210	0.9371 4318	0 9072 6777	0 8783 7991	0 8504 4606
27	0 9668 3355	0 9348 0646	0.9038 7823	0 8740 0986	0 8151 6378
28	0.9656 2652	0.9324 7527	0 9005 0135	0 8696 6155	0 8399 1432
29	0 9644 2100	0.9301 4990	0.8971 3709	0 8653 3188	0 8346 9746
30	0.9632 1697	0.9278 3032	0 8937 8539	0 8610 2973	0.8295 1300
31	0.9620 1446	0.9255 1653	0 8904 4622	0.8567 4600	0.8243 6075
32	0.9608 1344	0.9232 0851	0 8871 1952	0 8521 8358	0.8192 4050
33	0.9596 1392	0.9209 0624	0 8838 0525	0 8482 4237	0 8141 5205
34	0 9584 1590	0 9186 0972	0.8805 0330	0 8440 2226	0 8090 9520
35	0.9572 1938	0.9163 1892	0.8772 1381	0.8398 2314	0.8040 6976
36	0 9560 2435	0 9140 3384	0 8739 3655	0.8356 4492	0.7990 7554
37	0.9548 3081	0 9117 5445	0 8706 7153	0 8314 8748	0 7941 1234
38	0 9536 3876	0 9094 8075	0 8674 1871	0 8273 5073	0.7891 7997
39	0 9524 4820	0.9072 1272	0.8641 7804	0 8232 3455	0 7842 7823
40	0.9512 5913	0.9049 5034	0.8609 4948	0.8191 3886	0.7794 0693
41	0 9500 7154	0.9026 9361	0 8577 3298	0.8150 6354	0.7745 6590
42	0.9488 8543	0.9004 4250	0 8545 2850	0 8110 0850	0.7697 5493
43	0.9477 0080	0 8981 9701	0.8513 3599	0.8069 7363	0.7649 7384
44	0.9465 1766	0 8959 5712	0 8181 5541	0.8029 5884	0.7602 2245
45	0.9453 3599	0.8937 2281	0.8449 8671	0.7989 6402	0.7555 0057
46	0.9441 5579	0.8914 9407	0 8418 2985	0.7949 8907	0.7508 0802
47	0.9429 7707	0 8892 7090	0 8386 8478	0.7910 3390	0 7461 4462
48	0.9417 9982	0.8870 5326	0.8355 5146	0.7870 9841	0 7415 1018
49	0.9406 2404	0 8848 4116	0.8324 2985	0.7831 8250	0.7369 0453
50	0.9394 4973	0.8826 3457	0.8293 1990	0.7792.8607	0.7323 2748

PRESENT VALUE OF 1

$$v^n = \frac{1}{(1+i)^n}$$

n	¾%	1%	1⅛%	1¼%	1⅜%
1	0.9925 5583	0.9900 9901	0.9888 7515	0.9876 5432	0.9864 3650
2	0.9851 6708	0.9802 9605	0.9778 7407	0.9754 6106	0.9730 5696
3	0.9778 3333	0.9705 9015	0.9669 9537	0.9634 1833	0.9598 5890
4	0.9705 5417	0.9609 8034	0.9562 3770	0.9515 2428	0.9468 3986
5	0.9633 2920	0.9514 6569	0.9455 9970	0.9397 7706	0.9339 9739
6	0.9561 5802	0.9420 4524	0.9350 8005	0.9281 7488	0.9213 2912
7	0.9490 4022	0.9327 1805	0.9246 7743	0.9167 1593	0.9088 3267
8	0.9419 7540	0.9234 8322	0.9143 9054	0.9053 9845	0.8965 0571
9	0.9349 6318	0.9143 3982	0.9042 1808	0.8942 2069	0.8843 4596
10	0.9280 0315	0.9052 8695	0.8941 5880	0.8831 8093	0.8723 5113
11	0.9210 9494	0.8963 2372	0.8842 1142	0.8722 7746	0.8605 1899
12	0.9142 3815	0.8874 4923	0.8743 7470	0.8615 0860	0.8488 4734
13	0.9074 3211	0.8786 6260	0.8646 4742	0.8508 7269	0.8373 3400
14	0.9006 7733	0.8699 6297	0.8550 2835	0.8403 6809	0.8259 7682
15	0.8939 7254	0.8613 4947	0.8455 1629	0.8299 9318	0.8147 7368
16	0.8873 1766	0.8528 2126	0.8361 1005	0.8197 4635	0.8037 2250
17	0.8807 1231	0.8443 7749	0.8268 0846	0.8096 2602	0.7928 2120
18	0.8741 5614	0.8360 1731	0.8176 1034	0.7996 3064	0.7820 6777
19	0.8676 4878	0.8277 3992	0.8085 1455	0.7897 5866	0.7714 6020
20	0.8611 8985	0.8195 4447	0.7995 1995	0.7800 0855	0.7609 9649
21	0.8547 7901	0.8114 3017	0.7906 2542	0.7703 7881	0.7506 7472
22	0.8484 1589	0.8033 9621	0.7818 2983	0.7608 6796	0.7404 9294
23	0.8421 0014	0.7954 4179	0.7731 3210	0.7514 7453	0.7304 4926
24	0.8358 3140	0.7875 6613	0.7645 3112	0.7421 9707	0.7205 4181
25	0.8296 0933	0.7797 6844	0.7560 2583	0.7330 3414	0.7107 6874
26	0.8234 3358	0.7720 4796	0.7476 1516	0.7239 8434	0.7011 2823
27	0.8173 0380	0.7644 0392	0.7392 9806	0.7150 4626	0.6916 1847
28	0.8112 1966	0.7568 3557	0.7310 7348	0.7062 1853	0.6822 3771
29	0.8051 8080	0.7493 4215	0.7229 4040	0.6974 9978	0.6729 8417
30	0.7991 8690	0.7419 2292	0.7148 9780	0.6888 8867	0.6638 5615
31	0.7932 3762	0.7345 7715	0.7069 4467	0.6803 8387	0.6548 5194
32	0.7873 3262	0.7273 0411	0.6990 8002	0.6719 8407	0.6459 6985
33	0.7814 7158	0.7201 0307	0.6913 0287	0.6636 8797	0.6372 0824
34	0.7756 5418	0.7129 7334	0.6836 1223	0.6554 9429	0.6285 6546
35	0.7698 8008	0.7059 1420	0.6760 0715	0.6474 0177	0.6200 3991
36	0.7641 4896	0.6989 2495	0.6684 8667	0.6394 0916	0.6116 3000
37	0.7584 6051	0.6920 0490	0.6610 4986	0.6315 1522	0.6033 3416
38	0.7528 1440	0.6851 5337	0.6536 9578	0.6237 1873	0.5951 5083
39	0.7472 1032	0.6783 6967	0.6464 2352	0.6160 1850	0.5870 7850
40	0.7416 4796	0.6716 5314	0.6392 3216	0.6084 1334	0.5791 1566
41	0.7361 2701	0.6650 0311	0.6321 2080	0.6009 0206	0.5712 6083
42	0.7306 4716	0.6584 1892	0.6250 8855	0.5934 8352	0.5635 1253
43	0.7252 0809	0.6518 9992	0.6181 3454	0.5861 5656	0.5558 6933
44	0.7198 0952	0.6454 4546	0.6112 5789	0.5789 2006	0.5483 2979
45	0.7144 5114	0.6390 5492	0.6044 5774	0.5717 7290	0.5408 9252
46	0.7091 3264	0.6327 2764	0.5977 3324	0.5647 1397	0.5335 5612
47	0.7038 5374	0.6264 6301	0.5910 8355	0.5577 4219	0.5263 1923
48	0.6986 1414	0.6202 6041	0.5845 0784	0.5508 5649	0.5191 8050
49	0.6934 1353	0.6141 1921	0.5780 0528	0.5440 5579	0.5121 3860
50	0.6882 5165	0.6080 3882	0.5715 7506	0.5373 3905	0.5051 9220

$$v^n = \frac{1}{(1+i)^n}$$

n	1½%	1⅝%	1¾%	1⅞%	2%
1	0.9852 2167	0.9840 0984	0.9828 0098	0.9815 9509	0.9803 9216
2	0.9706 6175	0.9682 7537	0.9658 9777	0.9635 2892	0.9611 6878
3	0.9563 1699	0.9527 9249	0.9492 8528	0.9457 9526	0.9423 2233
4	0.9421 8423	0.9375 5718	0.9329 5851	0.9283 8799	0.9238 4543
5	0.9282 6033	0.9225 6549	0.9169 1254	0.9113 0109	0.9057 3081
6	0.9145 4219	0.9078 1352	0.9011 4254	0.8945 2868	0.8879 7138
7	0.9010 2679	0.8932 9744	0.8856 4378	0.8780 6496	0.8705 6018
8	0.8877 1112	0.8790 1347	0.8704 1157	0.8619 0426	0.8534 9037
9	0.8745 9224	0.8649 5791	0.8554 4135	0.8460 4099	0.8367 5527
10	0.8616 6723	0.8511 2709	0.8407 2860	0.8304 6968	0.8203 4830
11	0.8489 3323	0.8375 1743	0.8262 6889	0.8151 8496	0.8042 6304
12	0.8363 8742	0.8241 2539	0.8120 5788	0.8001 8156	0.7884 9318
13	0.8240 2702	0.8109 4750	0.7980 9128	0.7854 5429	0.7730 3253
14	0.8118 4928	0.7979 8032	0.7843 6490	0.7709 9808	0.7578 7502
15	0.7998 5150	0.7852 2048	0.7708 7459	0.7568 0793	0.7430 1473
16	0.7880 3104	0.7726 6468	0.7576 1631	0.7428 7895	0.7284 4581
17	0.7763 8526	0.7603 0965	0.7445 8605	0.7292 0633	0.7141 6256
18	0.7649 1159	0.7481 5218	0.7317 7990	0.7157 8536	0.7001 5937
19	0.7536 0747	0.7361 8911	0.7191 9401	0.7026 1139	0.6864 3076
20	0.7424 7042	0.7244 1732	0.7068 2458	0.6896 7989	0.6729 7133
21	0.7314 9795	0.7128 3378	0.6946 6789	0.6769 8640	0.6597 7582
22	0.7206 8763	0.7014 3545	0.6827 2028	0.6645 2653	0.6468 3904
23	0.7100 3708	0.6902 1938	0.6709 7817	0.6522 9598	0.6341 5592
24	0.6995 4392	0.6791 8267	0.6594 3800	0.6402 9053	0.6217 2149
25	0.6892 0583	0.6683 2243	0.6480 9632	0.6285 0604	0.6095 3087
26	0.6790 2052	0.6576 3584	0.6369 4970	0.6169 3844	0.5975 7928
27	0.6689 8574	0.6471 2014	0.6259 9479	0.6055 8375	0.5858 6204
28	0.6590 9925	0.6367 7259	0.6152 2829	0.5944 3804	0.5743 7455
29	0.6493 5887	0.6265 9049	0.6046 4697	0.5834 9746	0.5631 1231
30	0.6397 6243	0.6165 7121	0.5942 4764	0.5727 5824	0.5520 7089
31	0.6303 0781	0.6067 1214	0.5840 2716	0.5622 1668	0.5412 4597
32	0.6209 9292	0.5970 1071	0.5739 8247	0.5518 6913	0.5306 3330
33	0.6118 1568	0.5874 6442	0.5641 1053	0.5417 1203	0.5202 2873
34	0.6027 7407	0.5780 7077	0.5544 0839	0.5317 4187	0.5100 2817
35	0.5938 6608	0.5688 2732	0.5448 7311	0.5219 5521	0.5000 2761
36	0.5850 8974	0.5597 3168	0.5355 0183	0.5123 4867	0.4902 2315
37	0.5764 4309	0.5507 8148	0.5262 9172	0.5029 1894	0.4806 1093
38	0.5679 2423	0.5419 7440	0.5172 4002	0.4936 6277	0.4711 8719
39	0.5595 3126	0.5333 0814	0.5083 4400	0.4845 7695	0.4619 4822
40	0.5512 6232	0.5247 8046	0.4996 0098	0.4756 5836	0.4528 9042
41	0.5431 1559	0.5163 8914	0.4910 0834	0.4669 0391	0.4440 1021
42	0.5350 8925	0.5081 3199	0.4825 6348	0.4583 1058	0.4353 0413
43	0.5271 8153	0.5000 0688	0.4742 6386	0.4498 7542	0.4267 6875
44	0.5193 9067	0.4920 1169	0.4661 0699	0.4415 9550	0.4184 0074
45	0.5117 1494	0.4841 4434	0.4580 9040	0.4334 6798	0.4101 9680
46	0.5041 5265	0.4764 0280	0.4502 1170	0.4254 9004	0.4021 5373
47	0.4967 0212	0.4687 8504	0.4424 6850	0.4176 5894	0.3942 6836
48	0.4893 6170	0.4612 8909	0.4348 5848	0.4099 7196	0.3865 3761
49	0.4821 2975	0.4539 1301	0.4273 7934	0.4024 2647	0.3789 5844
50	0.4750 0468	0.4466 5487	0.4200 2883	0.3950 1984	0.3715 2788

$$v^n = \frac{1}{(1+i)^n}$$

n	2⅛%	2¼%	2⅜%	2½%	2¾%
1	0.9791 9217	0.9779 9511	0.9768 0098	0.9756 0976	0.9732 3601
2	0.9588 1730	0.9564 7444	0.9541 4015	0.9518 1440	0.9471 8833
3	0.9388 6639	0.9354 2732	0.9320 0503	0.9285 9941	0.9218 3779
4	0.9193 3061	0.9148 4335	0.9103 8342	0.9059 5064	0.8971 6573
5	0.9002 0133	0.8947 1232	0.8892 6342	0.8838 5429	0.8731 5400
6	0.8814 7009	0.8750 2427	0.8686 3337	0.8622 9687	0.8497 8491
7	0.8631 2861	0.8557 6946	0.8484 8193	0.8412 6524	0.8270 4128
8	0.8451 6878	0.8369 3835	0.8287 9798	0.8207 4657	0.8049 0635
9	0.8275 8264	0.8185 2161	0.8095 7067	0.8007 2836	0.7833 6385
10	0.8103 6244	0.8005 1013	0.7907 8942	0.7811 9840	0.7623 9791
11	0.7935 0056	0.7828 9499	0.7724 4388	0.7621 4478	0.7419 9310
12	0.7769 8953	0.7656 6748	0.7545 2394	0.7435 5589	0.7221 3440
13	0.7608 2206	0.7488 1905	0.7370 1972	0.7254 2038	0.7028 0720
14	0.7449 9100	0.7323 4137	0.7199 2158	0.7077 2720	0.6839 9728
15	0.7294 8935	0.7162 2628	0.7032 2010	0.6904 6556	0.6656 9078
16	0.7143 1026	0.7004 6580	0.6869 0609	0.6736 2493	0.6478 7424
17	0.6994 4701	0.6850 5212	0.6709 7053	0.6571 9506	0.6305 3454
18	0.6848 9303	0.6699 7763	0.6554 0467	0.6411 6591	0.6136 5892
19	0.6706 4189	0.6552 3484	0.6401 9993	0.6255 2772	0.5972 3496
20	0.6566 8729	0.6408 1647	0.6253 4791	0.6102 7094	0.5812 5057
21	0.6430 2305	0.6267 1538	0.6108 4045	0.5953 8629	0.5656 9398
22	0.6296 4313	0.6129 2457	0.5966 6955	0.5808 6467	0.5505 5375
23	0.6165 4162	0.5994 3724	0.5828 2740	0.5666 9724	0.5358 1874
24	0.6037 1273	0.5862 4668	0.5693 0637	0.5528 7535	0.5214 7809
25	0.5911 5077	0.5733 4639	0.5560 9902	0.5393 9059	0.5075 2126
26	0.5788 5021	0.5607 2997	0.5431 9807	0.5262 3472	0.4939 3796
27	0.5668 0559	0.5483 9117	0.5305 9640	0.5133 9973	0.4807 1821
28	0.5550 1159	0.5363 2388	0.5182 8708	0.5008 7778	0.4678 5227
29	0.5434 6300	0.5245 2213	0.5062 6333	0.4886 6125	0.4553 3068
30	0.5321 5471	0.5129 8008	0.4945 1852	0.4767 4269	0.4431 4421
31	0.5210 8173	0.5016 9201	0.4830 4617	0.4651 1481	0.4312 8391
32	0.5102 3915	0.4906 5233	0.4718 3997	0.4537 7055	0.4197 4103
33	0.4996 2217	0.4798 5558	0.4608 9374	0.4427 0298	0.4085 0708
34	0.4892 2612	0.4692 9641	0.4502 0146	0.4319 0534	0.3975 7380
35	0.4790 4638	0.4589 6960	0.4397 5722	0.4213 7107	0.3869 3314
36	0.4690 7846	0.4488 7002	0.4295 5529	0.4110 9372	0.3765 7727
37	0.4593 1796	0.4389 9268	0.4195 9002	0.4010 6705	0.3664 9856
38	0.4497 6055	0.4293 3270	0.4098 5594	0.3912 8492	0.3566 8959
39	0.4404 0201	0.4198 8528	0.4003 4769	0.3817 4139	0.3471 4316
40	0.4312 3819	0.4106 4575	0.3910 6001	0.3724 3062	0.3378 5222
41	0.4222 6506	0.4016 0954	0.3819 8780	0.3633 4695	0.3288 0995
42	0.4134 7864	0.3927 7216	0.3731 2606	0.3544 8483	0.3200 0968
43	0.4048 7505	0.3841 2925	0.3644 6990	0.3458 3886	0.3114 4495
44	0.3964 5047	0.3756 7653	0.3560 1455	0.3374 0376	0.3031 0944
45	0.3882 0120	0.3674 0981	0.3477 5536	0.3291 7440	0.2949 9702
46	0.3801 2357	0.3593 2500	0.3396 8778	0.3211 4576	0.2871 0172
47	0.3722 1402	0.3514 1809	0.3318 0735	0.3133 1294	0.2794 1773
48	0.3644 6906	0.3436 8518	0.3241 0975	0.3056 7116	0.2719 3940
49	0.3568 8524	0.3361 2242	0.3165 9072	0.2982 1576	0.2646 6122
50	0.3494 5924	0.3287 2608	0.3092 4612	0.2909 4221	0.2575 7783

$$v^n = \frac{1}{(1+i)^n}$$

n	3%	3¼%	3½%	3¾%	4%
1	0.9708 7379	0.9685 2300	0.9661 8357	0.9638 5542	0.9615 3846
2	0.9425 9591	0.9380 3681	0.9335 1070	0.9290 1727	0.9245 5621
3	0.9151 4166	0.9085 1022	0.9019 4271	0.8954 3834	0.8889 9636
4	0.8884 8705	0.8799 1305	0.8714 4223	0.8630 7310	0.8548 0419
5	0.8626 0878	0.8522 1603	0.8419 7317	0.8318 7768	0.8219 2711
6	0.8374 8426	0.8253 9083	0.8135 0064	0.8018 0981	0.7903 1453
7	0.8130 9151	0.7994 1000	0.7859 9096	0.7728 2874	0.7599 1781
8	0.7894 0923	0.7742 4698	0.7594 1156	0.7448 9517	0.7306 9021
9	0.7664 1673	0.7498 7601	0.7337 3097	0.7179 7125	0.7025 8674
10	0.7440 9391	0.7262 7216	0.7089 1881	0.6920 2048	0.6755 6417
11	0.7224 2128	0.7034 1129	0.6849 4571	0.6670 0769	0.6495 8093
12	0.7013 7988	0.6812 7002	0.6617 8330	0.6428 9898	0.6245 9705
13	0.6809 5134	0.6598 2568	0.6394 0415	0.6196 6167	0.6005 7409
14	0.6611 1781	0.6390 5635	0.6177 8179	0.5972 6426	0.5774 7508
15	0.6418 6195	0.6189 4078	0.5968 9062	0.5756 7639	0.5552 6450
16	0.6231 6694	0.5994 5838	0.5767 0591	0.5548 6881	0.5339 0818
17	0.6050 1645	0.5805 8923	0.5572 0378	0.5348 1331	0.5133 7325
18	0.5873 9461	0.5623 1402	0.5383 6114	0.5154 8271	0.4936 2812
19	0.5702 8603	0.5446 1407	0.5201 5569	0.4968 5080	0.4746 4242
20	0.5536 7575	0.5274 7125	0.5025 6588	0.4788 9234	0.4563 8695
21	0.5375 4928	0.5108 6804	0.4855 7090	0.4615 8298	0.4388 3360
22	0.5218 9250	0.4947 8745	0.4691 5063	0.4448 9925	0.4219 5539
23	0.5066 9175	0.4792 1302	0.4532 8563	0.4288 1856	0.4057 2633
24	0.4919 3374	0.4641 2884	0.4379 5713	0.4133 1910	0.3901 2147
25	0.4776 0557	0.4495 1945	0.4231 4699	0.3983 7985	0.3751 1680
26	0.4636 9473	0.4353 6993	0.4088 3767	0.3839 8058	0.3606 8923
27	0.4501 8906	0.4216 6579	0.3950 1224	0.3701 0176	0.3468 1657
28	0.4370 7675	0.4083 9302	0.3816 5434	0.3567 2459	0.3334 7747
29	0.4243 4636	0.3955 3803	0.3687 4815	0.3438 3093	0.3206 5141
30	0.4119 8676	0.3830 8768	0.3562 7841	0.3314 0331	0.3083 1867
31	0.3999 8715	0.3710 2923	0.3442 3035	0.3194 2187	0.2964 6026
32	0.3883 3703	0.3593 5035	0.3325 8971	0.3078 7940	0.2850 5794
33	0.3770 2625	0.3480 3908	0.3213 4271	0.2967 5123	0.2740 9417
34	0.3660 4490	0.3370 8385	0.3104 7605	0.2860 2528	0.2635 5209
35	0.3553 8340	0.3264 7346	0.2999 7686	0.2756 8702	0.2534 1547
36	0.3450 3243	0.3161 9706	0.2898 3272	0.2657 2242	0.2436 6872
37	0.3349 8294	0.3062 4413	0.2800 3161	0.2561 1800	0.2342 9685
38	0.3252 2615	0.2966 0448	0.2705 6194	0.2468 6072	0.2252 8543
39	0.3157 5355	0.2872 6826	0.2614 1250	0.2379 3805	0.2166 2061
40	0.3065 5684	0.2782 2592	0.2525 7247	0.2293 3788	0.2082 8904
41	0.2976 2800	0.2694 6820	0.2440 3137	0.2210 4855	0.2002 7793
42	0.2889 5922	0.2609 8615	0.2357 7910	0.2130 5885	0.1925 7493
43	0.2805 4294	0.2527 7109	0.2278 0590	0.2053 5793	0.1851 6820
44	0.2723 7178	0.2448 1462	0.2201 0231	0.1979 3535	0.1780 4635
45	0.2644 3862	0.2371 0859	0.2126 5924	0.1907 8106	0.1711 9841
46	0.2567 3653	0.2296 4512	0.2054 6787	0.1838 8536	0.1646 1386
47	0.2492 5876	0.2224 1658	0.1985 1968	0.1772 3890	0.1582 8256
48	0.2419 9880	0.2154 1558	0.1918 0645	0.1708 3268	0.1521 9476
49	0.2349 5029	0.2086 3494	0.1853 2024	0.1646 5800	0.1463 4112
50	0.2281 0708	0.2020 6774	0.1790 5337	0.1587 0651	0.1407 1262

PRESENT VALUE OF 1

$$v^n = \frac{1}{(1+i)^n}$$

n	4¼%	4½%	4¾%	5%	5½%
1	0.9592 3261	0.9569 3780	0.9546 5394	0.9523 8095	0.9478 6730
2	0.9201 2721	0.9157 2995	0.9113 6414	0.9070 2948	0.8984 5242
3	0.8826 1603	0.8762 9660	0.8700 3737	0.8638 3760	0.8516 1366
4	0.8466 3408	0.8385 6134	0.8305 8460	0.8227 0247	0.8072 1674
5	0.8121 1902	0.8024 5105	0.7929 2086	0.7835 2617	0.7651 3135
6	0.7790 1105	0.7678 9574	0.7569 6502	0.7462 1540	0.7252 4583
7	0.7472 5281	0.7348 2846	0.7226 3964	0.7106 8133	0.6874 3681
8	0.7167 8926	0.7031 8513	0.6898 7077	0.6768 3936	0.6515 9887
9	0.6875 6764	0.6729 0443	0.6585 8785	0.6446 0892	0.6176 2926
10	0.6595 3730	0.6439 2768	0.6287 2349	0.6139 1325	0.5854 3058
11	0.6326 4969	0.6161 9874	0.6002 1335	0.5846 7929	0.5549 1050
12	0.6068 5822	0.5896 6386	0.5729 9604	0.5568 3742	0.5259 8152
13	0.5821 1819	0.5642 7164	0.5470 1293	0.5303 2135	0.4985 6068
14	0.5583 8676	0.5399 7286	0.5222 0804	0.5050 6795	0.4725 6937
15	0.5356 2279	0.5167 2044	0.4985 2797	0.4810 1710	0.4479 3305
16	0.5137 8685	0.4944 6932	0.4759 2169	0.4581 1152	0.4245 8109
17	0.4928 4110	0.4731 7639	0.4543 4051	0.4362 9669	0.4024 4653
18	0.4727 4926	0.4528 0037	0.4337 3796	0.4155 2065	0.3814 6590
19	0.4534 7650	0.4333 0179	0.4140 6965	0.3957 3396	0.3615 7906
20	0.4349 8945	0.4146 4286	0.3952 9322	0.3768 8948	0.3427 2896
21	0.4172 5607	0.3967 8743	0.3773 6823	0.3589 4236	0.3248 6158
22	0.4002 4563	0.3797 0089	0.3602 5607	0.3418 4987	0.3079 2567
23	0.3839 2866	0.3633 5013	0.3439 1987	0.3255 7131	0.2918 7267
24	0.3682 7689	0.3477 0347	0.3283 2446	0.3100 6791	0.2766 5656
25	0.3532 6321	0.3327 3060	0.3134 3624	0.2953 0277	0.2622 3370
26	0.3388 6159	0.3184 0248	0.2992 2314	0.2812 4073	0.2485 6275
27	0.3250 4709	0.3046 9137	0.2856 5455	0.2678 4832	0.2356 0450
28	0.3117 9577	0.2915 7069	0.2727 0124	0.2550 9364	0.2233 2181
29	0.2990 8467	0.2790 1502	0.2603 3531	0.2429 4632	0.2116 7944
30	0.2868 9177	0.2670 0002	0.2485 3013	0.2313 7745	0.2006 4402
31	0.2751 9594	0.2555 0241	0.2372 6027	0.2203 5947	0.1901 8390
32	0.2639 7692	0.2444 9991	0.2265 0145	0.2098 6617	0.1802 6910
33	0.2532 1527	0.2339 7121	0.2162 3050	0.1998 7254	0.1708 7119
34	0.2428 9235	0.2238 9589	0.2064 2530	0.1903 5480	0.1619 6321
35	0.2329 9026	0.2142 5444	0.1970 6473	0.1812 9029	0.1535 1963
36	0.2234 9186	0.2050 2817	0.1881 2862	0.1726 5741	0.1455 1624
37	0.2143 8068	0.1961 9921	0.1795 9772	0.1644 3563	0.1379 3008
38	0.2056 4094	0.1877 5044	0.1714 5367	0.1566 0536	0.1307 3941
39	0.1972 5750	0.1796 6549	0.1636 7893	0.1491 4797	0.1239 2362
40	0.1892 1582	0.1719 2870	0.1562 5673	0.1420 4568	0.1174 6314
41	0.1815 0199	0.1645 2507	0.1491 7110	0.1352 8160	0.1113 3947
42	0.1741 0263	0.1574 4026	0.1424 0678	0.1288 3962	0.1055 3504
43	0.1670 0492	0.1506 6054	0.1359 4919	0.1227 0440	0.1000 3322
44	0.1601 9657	0.1441 7276	0.1297 8443	0.1168 6133	0.0948 1822
45	0.1536 6577	0.1379 6437	0.1238 9922	0.1112 9651	0.0898 7509
46	0.1474 0122	0.1320 2332	0.1182 8088	0.1059 9668	0.0851 8965
47	0.1413 9206	0.1263 3810	0.1129 1731	0.1009 4921	0.0807 4849
48	0.1356 2787	0.1208 9771	0.1077 9695	0.0961 4211	0.0765 3885
49	0.1300 9868	0.1156 9158	0.1029 0878	0.0915 6391	0.0725 4867
50	0.1247 9489	0.1107 0965	0.0982 4228	0.0872 0373	0.0687 6652

$$v^n = \frac{1}{(1+i)^n}$$

n	6%	6½%	7%	8%	9%
1	0.9433 9623	0.9389 6714	0.9345 7944	0.9259 2593	0.9174 3119
2	0.8899 9644	0.8816 5928	0.8734 3873	0.8573 3882	0.8416 7999
3	0.8396 1928	0.8278 4909	0.8162 9788	0.7938 3224	0.7721 8348
4	0.7920 9366	0.7773 2309	0.7628 9521	0.7350 2985	0.7084 2521
5	0.7472 5817	0.7298 8084	0.7129 8618	0.6805 8320	0.6499 3139
6	0.7049 6054	0.6853 3412	0.6663 4222	0.6301 6963	0.5962 6733
7	0.6650 5711	0.6435 0621	0.6227 4974	0.5834 9040	0.5470 3424
8	0.6274 1237	0.6042 3119	0.5820 0910	0.5402 6888	0.5018 6628
9	0.5918 9846	0.5673 5323	0.5439 3374	0.5002 4897	0.4604 2778
10	0.5583 9478	0.5327 2604	0.5083 4929	0.4631 9349	0.4224 1081
11	0.5267 8753	0.5002 1224	0.4750 9280	0.4288 8286	0.3875 3285
12	0.4969 6936	0.4696 8285	0.4440 1196	0.3971 1376	0.3555 3473
13	0.4688 3902	0.4410 1676	0.4149 6445	0.3676 9792	0.3261 7865
14	0.4423 0096	0.4141 0025	0.3878 1724	0.3404 6104	0.2992 4647
15	0.4172 6506	0.3888 2652	0.3624 4602	0.3152 4170	0.2745 3804
16	0.3936 4628	0.3650 9533	0.3387 3460	0.2918 9047	0.2518 6976
17	0.3713 6442	0.3428 1251	0.3165 7439	0.2702 6895	0.2310 7318
18	0.3503 4379	0.3218 8969	0.2958 6392	0.2502 4903	0.2119 9374
19	0.3305 1301	0.3022 4384	0.2765 0832	0.2317 1206	0.1944 8967
20	0.3118 0473	0.2837 9703	0.2584 1900	0.2145 4821	0.1784 3089
21	0.2941 5540	0.2664 7608	0.2415 1309	0.1986 5575	0.1636 9806
22	0.2775 0510	0.2502 1228	0.2257 1317	0.1839 4051	0.1501 8171
23	0.2617 9726	0.2349 4111	0.2109 4688	0.1703 1528	0.1377 8139
24	0.2469 7855	0.2206 0198	0.1971 4662	0.1576 9934	0.1264 0494
25	0.2329 9863	0.2071 3801	0.1842 4918	0.1460 1790	0.1159 6784
26	0.2198 1003	0.1944 9579	0.1721 9549	0.1352 0176	0.1063 9251
27	0.2073 6795	0.1826 2515	0.1609 3037	0.1251 8682	0.0976 0781
28	0.1956 3014	0.1714 7902	0.1504 0221	0.1159 1372	0.0895 4845
29	0.1845 5674	0.1610 1316	0.1405 6282	0.1073 2752	0.0821 5454
30	0.1741 1013	0.1511 8607	0.1313 6712	0.0993 7733	0.0753 7114
31	0.1642 5484	0.1419 5875	0.1227 7301	0.0920 1605	0.0691 4783
32	0.1549 5740	0.1332 9460	0.1147 4113	0.0852 0005	0.0634 3838
33	0.1461 8622	0.1251 5925	0.1072 3470	0.0788 8893	0.0582 0035
34	0.1379 1153	0.1175 2042	0.1002 1934	0.0730 4531	0.0533 9481
35	0.1301 0522	0.1103 4781	0.0936 6294	0.0676 3454	0.0489 8607
36	0.1227 4077	0.1036 1297	0.0875 3546	0.0626 2458	0.0449 4135
37	0.1157 9318	0.0972 8917	0.0818 0884	0.0579 8572	0.0412 3059
38	0.1092 3885	0.0913 5134	0.0764 5686	0.0536 9048	0.0378 2623
39	0.1030 5552	0.0857 7590	0.0714 5501	0.0497 1341	0.0347 0296
40	0.0972 2219	0.0805 4075	0.0667 8038	0.0460 3093	0.0318 3758
41	0.0917 1905	0.0756 2512	0.0624 1157	0.0426 2123	0.0292 0879
42	0.0865 2740	0.0710 0950	0.0583 2857	0.0394 6411	0.0267 9706
43	0.0816 2962	0.0666 7559	0.0545 1268	0.0365 4084	0.0245 8446
44	0.0770 0908	0.0626 0619	0.0509 4643	0.0338 3411	0.0225 5455
45	0.0726 5007	0.0587 8515	0.0476 1349	0.0313 2788	0.0206 9224
46	0.0685 3781	0.0551 9733	0.0444 9859	0.0290 0730	0.0189 8371
47	0.0646 5831	0.0518 2848	0.0415 8747	0.0268 5861	0.0174 1625
48	0.0609 9840	0.0486 6524	0.0388 6679	0.0248 6908	0.0159 7821
49	0.0575 4566	0.0456 9506	0.0363 2410	0.0230 2693	0.0146 5891
50	0.0542 8836	0.0429 0616	0.0339 4776	0.0213 2123	0.0134 4854

Table 4

AMOUNT OF ANNUITY OF 1

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

n	½%	1%	1¼%	1½%	1¾%	2%
1	1.000 0000	1 000 0000	1 000 0000	1.000 0000	1.000 0000	1.000 0000
2	2 005 0000	2 010 0000	2 012 5000	2.015 0000	2 017 5000	2.020 0000
3	3.015 0250	3.030 1000	3 037 6562	3 045 2250	3 052 8063	3 060 4000
4	4.030 1001	4.060 4010	4.075 6270	4.090 9034	4.106 2304	4 121 6080
5	5.050 2506	5.101 0050	5.126 5723	5.152 2669	5.178 0894	5.204 0402
6	6.075 5019	6.152 0151	6.190 6544	6.229 5509	6.268 7060	6.308 1210
7	7.105 8794	7.213 5352	7.268 0376	7 322 9942	7 378 4083	7.434 2834
8	8.141 4088	8.285 6706	8.358 8881	8 432 8391	8 507 5305	8 582 9691
9	9.182 1158	9.368 5273	9.463 3742	9.559 3317	9.656 4122	9 754 6284
10	10.228 0264	10.462 2125	10.581 6664	10.702 7217	10.825 3995	10.949 7210
11	11 279 1665	11.566 8347	11.713 9372	11 863 2625	12.014 8439	12.168 7154
12	12 335 5624	12.682 5030	12.860 3614	13 041 2114	13.225 1037	13.412 0897
13	13.397 2402	13 809 3280	14.021 1159	14 236 8296	14 456 5430	14 680 3315
14	14.464 2264	14 947 4213	15.196 3799	15.450 3821	15.709 5325	15.973 9382
15	15 536 5475	16 096 8955	16.386 3346	16.682 1378	16.984 4494	17.293 4169
16	16 614 2303	17 257 8645	17 591 1638	17.932 3698	18.281 6772	18.639 2853
17	17.697 3014	18.430 4431	18 811 0534	19.201 3554	19 601 6066	20.012 0710
18	18.785 7879	19 614 7476	20 046 1915	20 489 3757	20.944 6347	21.412 3124
19	19 879 7168	20 810 8950	21 296 7689	21.796 7164	22 311 1658	22.840 5586
20	20.979 1154	22.019 0040	22.562 9785	23.123 6671	23.701 6112	24.297 3698
21	22 084 0110	23 239 1940	23 845 0158	24.470 5221	25.116 3894	25.783 3172
22	23 194 4311	24 471 5860	25 143 0785	25 837 5799	26 555 9262	27 298 9835
23	24 310 4032	25 716 3018	26 457 3670	27.225 1436	28.020 6549	28.844 9632
24	25.431 9552	26.973 4648	27 788 0840	28 633 5208	29 511 0164	30.421 8625
25	26.559 1150	28 243 1995	29.135 4351	30 063 0236	31.027 4592	32.030 2907
26	27 691 9106	29 525 6315	30.499 6280	31 513 9690	32.570 4397	33.670 9057
27	28.830 3702	30.820 8878	31.880 8734	32 986 6785	34.140 4224	35.344 3238
28	29.974 5220	32.129 0967	33 279 3843	34.481 4787	35.737 8798	37.051 2103
29	31 124 3946	33 450 3877	34 695 3766	35 998 7009	37 363 2927	38.792 2345
30	32.280 0166	34.784 8915	36.129 0688	37.538 6814	39.017 1503	40.568 0792
31	33.441 4167	36.132 7404	37.580 6822	39.101 7616	40.699 9504	42.379 4408
32	34 608 6238	37.494 0678	39 050 4407	40.688 2880	42.412 1996	44 227 0296
33	35 781 6669	38 869 0085	40.538 5712	42 298 6123	44.154 4131	46 111 5702
34	36 960 5752	40.257 6986	42.045 3033	43.933 0915	45.927 1153	48.033 8016
35	38.145 3781	41.660 2756	43.570 8696	45 592 0879	47.730 8398	49 994 4776
36	39.336 1050	43 076 8784	45 115 5055	47.275 9692	49.566 1295	51 994 3672
37	40.532 7855	44 507 6471	46 679 4493	48.985 1087	51.433 5368	54 034 2545
38	41 735 4494	45.952 7236	48 262 9424	50 719 8854	53.333 6236	56 114 9396
39	42.944 1267	47 412 2508	49.866 2292	52 480 6837	55 266 9621	58 237 2384
40	44.158 8473	48.886 3734	51.489 5571	54.267 8939	57.234 1339	60.401 9832
41	45.379 6415	50.375 2371	53 133 1765	56.081 9123	59 235 7312	62.610 0228
42	46.606 5397	51.878 9895	54 797 3412	57.923 1410	61 272 3565	64.862 2233
43	47.839 5724	53 397 7794	56 482 3080	59 791 9881	63 344 6228	67 159 4678
44	49.078 7703	54.931 7572	58.188 3369	61.688 8679	65.453 1537	69.502 6571
45	50.324 1642	56.481 0747	59.915 6911	63.614 2010	67.598 5839	71.892 7103
46	51.575 7850	58.045 8855	61.664 6372	65.568 4140	69.781 5591	74.330 5645
47	52 833 6639	59.626 3443	63.435 4452	67.551 9402	72 002 7364	76 817 1758
48	54.097 8322	61.222 6078	65.228 3882	69.565 2193	74.262 7843	79.353 5193
49	55.368 3214	62.834 8338	67.043 7431	71 608 6976	76.562 3830	81.940 5897
50	56.645 1630	64.463 1822	68.881 7899	73.682 8280	78.902 2247	84.579 4015

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

n	2¼%	2½%	2¾%	3%	3½%	4%
1	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1.000 0000
2	2.022 5000	2.025 0000	2.027 5000	2.030 0000	2.035 0000	2.040 0000
3	3.068 0063	3.075 6250	3.083 2563	3.090 9000	3.106 2250	3.121 6000
4	4.137 0364	4.152 5156	4.168 0458	4.183 6270	4.214 9429	4.246 4640
5	5.230 1197	5.256 3285	5.282 6671	5.309 1358	5.362 4659	5.416 3226
6	6.347 7974	6.387 7367	6.427 9404	6.468 4099	6.550 1522	6.632 9755
7	7.490 6228	7.547 4302	7.604 7088	7.662 4622	7.779 4075	7.898 2945
8	8.659 1619	8.736 1159	8.813 8383	8.892 3361	9.051 6868	9.214 2263
9	9.853 9930	9.954 5188	10.056 2188	10.159 1061	10.368 4958	10.582 7953
10	11.075 7078	11.203 3818	11.332 7648	11.463 8793	11.731 3932	12.006 1071
11	12.324 9113	12.483 4663	12.644 4159	12.807 7957	13.141 9919	13.486 3514
12	13.602 2218	13.795 5530	13.992 1373	14.192 0296	14.601 9616	15.025 8055
13	14.908 2718	15.140 4418	15.376 9211	15.617 7905	16.113 0303	16.626 8377
14	16.243 7079	16.518 9528	16.799 7864	17.086 3242	17.676 9864	18.291 9112
15	17.609 1913	17.931 9267	18.261 7805	18.598 9139	19.295 6809	20.023 5876
16	19.005 3981	19.380 2248	19.763 9795	20.156 8813	20.971 0297	21.824 5311
17	20.433 0196	20.864 7305	21.307 4889	21.761 5877	22.705 0158	23.697 5124
18	21.892 7625	22.386 3487	22.893 4449	23.414 4354	24.499 6913	25.645 4129
19	23.385 3497	23.946 0074	24.523 0146	25.116 8684	26.357 1805	27.671 2294
20	24.911 5200	25.544 6576	26.197 3975	26.870 3745	28.279 6818	29.778 0786
21	26.472 0292	27.183 2741	27.917 8259	28.676 4857	30.269 4707	31.969 2017
22	28.067 6499	28.862 8559	29.685 5662	30.536 7803	32.328 9022	34.247 9698
23	29.699 1720	30.584 4273	31.501 9192	32.452 8837	34.460 4137	36.617 8886
24	31.367 4034	32.349 0380	33.368 2220	34.426 4702	36.666 5282	39.082 6041
25	33.073 1700	34.157 7639	35.285 8481	36.459 2643	38.949 8567	41.645 9083
26	34.817 3163	36.011 7080	37.256 2089	38.553 0423	41.313 1017	44.311 7446
27	36.600 7059	37.912 0007	39.280 7547	40.709 6335	43.759 0602	47.084 2144
28	38.424 2218	39.859 8008	41.360 9754	42.930 9225	46.290 6273	49.967 5830
29	40.288 7668	41.856 2958	43.498 4022	45.218 8502	48.910 7993	52.966 2863
30	42.195 2640	43.902 7032	45.694 6083	47.575 4157	51.622 6773	56.084 9378
31	44.144 6575	46.000 2707	47.951 2100	50.002 6782	54.429 4710	59.328 3353
32	46.137 9123	48.150 2775	50.269 8683	52.502 7585	57.334 5025	62.701 4687
33	48.176 0153	50.354 0345	52.652 2897	55.077 8413	60.341 2101	66.209 5274
34	50.259 9756	52.612 8853	55.100 2277	57.730 1765	63.453 1521	69.857 9085
35	52.390 8251	54.928 2074	57.615 4839	60.462 0818	66.674 0127	73.652 2249
36	54.569 6186	57.301 4126	60.199 9097	63.275 9443	70.007 6032	77.598 3139
37	56.797 1351	59.733 9479	62.855 4072	66.174 2226	73.457 8693	81.702 2464
38	59.075 3774	62.227 2966	65.583 9309	69.159 4193	77.028 8947	85.970 3363
39	61.404 5733	64.782 9791	68.387 4890	72.234 2328	80.724 9060	90.409 1497
40	63.786 1762	67.402 5535	71.268 1450	75.401 2597	84.550 2778	95.025 5157
41	66.221 3652	70.087 6174	74.228 0190	78.663 2975	88.509 5375	99.826 5363
42	68.711 3459	72.839 8078	77.269 2895	82.023 1965	92.607 3713	104.819 5978
43	71.257 3512	75.660 8030	80.394 1950	85.483 8923	96.848 6293	110.012 3817
44	73.860 6416	78.552 3231	83.605 0353	89.048 4091	101.238 3313	115.412 8770
45	76.522 5061	81.516 1312	86.904 1738	92.719 8614	105.781 6729	121.029 3920
46	79.244 2624	84.554 0344	90.294 0386	96.501 4572	110.484 0315	126.870 5677
47	82.027 2583	87.667 8853	93.777 1246	100.396 5010	115.350 9726	132.945 3904
48	84.872 8717	90.859 5824	97.355 9956	104.408 3960	120.388 2566	139.263 2060
49	87.782 5113	94.131 0720	101.033 2854	108.540 6479	125.601 8456	145.833 7343
50	90.757 6178	97.484 3488	104.811 7008	112.796 8673	130.997 9102	152.667 0837

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

n	4½%	5%	5½%	6%	7%	8%
1	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1.000 0000
2	2.045 0000	2.050 0000	2.055 0000	2.060 0000	2.070 0000	2.080 0000
3	3.137 0250	3.152 5000	3.168 0250	3.183 6000	3 214 9000	3.246 4000
4	4.278 1911	4 310 1250	4.342 2664	4.374 6160	4.439 9430	4.506 1120
5	5.470 7097	5.525 6313	5.581 0910	5.637 0930	5.750 7390	5.866 6010
6	6.716 8917	6 801 9128	6 888 0510	6.975 3185	7.153 2907	7.335 9290
7	8.019 1518	8.142 0085	8.266 8938	8.393 8377	8.654 0211	8.922 8034
8	9.380 0136	9.549 1089	9.721 5730	9.897 4679	10.259 8026	10.630 6276
9	10 802 1142	11 026 5643	11 256 2595	11.491 3160	11.977 9888	12.487 5578
10	12.288 2094	12.577 8925	12.875 3538	13.180 7949	13.816 4480	14.486 5625
11	13.841 1788	14.206 7872	14 583 4983	14.971 6426	15.783 5993	16.645 4875
12	15.464 0318	15.917 1265	16.385 5907	16 869 9412	17.888 4513	18.977 1265
13	17.159 9133	17.712 9829	18 286 7981	18.882 1377	20 140 6429	21.495 2966
14	18.932 1094	19.598 6320	20.292 5720	21.015 0659	22.550 4879	24.214 9203
15	20.784 0543	21.578 5636	22.408 0635	23.275 9699	25.129 0220	27.152 1139
16	22 719 3367	23.657 4918	24 641 1400	25 672 5281	27 888 0536	30.324 2830
17	24 741 7069	25 840 3664	26.996 4027	28 212 8798	30.840 2173	33.750 2257
18	26 855 0837	28.132 3847	29.481 2048	30 905 6526	33.999 0325	37.450 2437
19	29.063 5625	30 539 0039	32 102 6711	33.759 9917	37 378 9648	41.446 2632
20	31.371 4228	33.065 9541	34.868 3180	36.785 5912	40.995 4923	45.761 9643
21	33 783 1368	35.719 2518	37.786 0755	39.992 7267	44.865 1768	50.422 9214
22	36.303 3780	38 505 2144	40 864 3097	43 392 2903	49.005 7392	55.456 7552
23	38.937 0300	41 430 4751	44 111 8467	46.995 8277	53 436 1409	60.893 2956
24	41.689 1963	44 501 9989	47 537 9983	50.815 5774	58.176 6708	66.764 7592
25	44.565 2102	47.727 0988	51.152 5882	54.864 5120	63.249 0377	73.105 9400
26	47.570 6146	51.113 4538	54.965 9805	59.156 3827	68.676 4704	79.954 4152
27	50.711 3236	54 669 1265	58 989 1094	63.705 7657	74.483 8233	87.350 7684
28	53.993 3332	58 402 5828	63 233 5105	68 528 1116	80.697 6909	95.338 8298
29	57.423 0332	62.322 7119	67.711 3535	73.639 7983	87.346 5293	103.965 9362
30	61.007 0697	66.438 8475	72.435 4780	79.058 1862	94.460 7863	113.283 2111
31	64.752 3878	70 760 7899	77.419 4293	84.801 6774	102 073 0414	123.345 8680
32	68.666 2452	75.298 8294	82.677 4979	90.889 7780	110 218 1543	134.213 5374
33	72 756 2263	80 063 7708	88 224 7603	97.343 1647	118.933 4251	145 950 6204
34	77.030 2565	85 066 9594	94.077 1221	104.183 7546	128.258 7648	158.626 6701
35	81.496 6180	90.320 3074	100.251 3638	111.434 7799	138.236 8784	172.316 8037
36	86.163 9658	95 836 3227	106 765 1888	119.120 8667	148.913 4598	187.102 1480
37	91 041 3443	101.628 1389	113 637 2742	127.268 1187	160.337 4020	203.070 3198
38	96.138 2048	107.709 5458	120 887 3243	135 904 2058	172 561 0202	220.315 9454
39	101.464 4240	114 095 0231	128.536 1271	145.058 4581	185.640 2916	238.941 2210
40	107.030 3231	120.799 7742	136.605 6141	154.761 9656	199.635 1120	259.056 5187
41	112.846 6876	127 839 7630	145.118 9229	165.047 6836	214.609 5698	280.781 0402
42	118.924 7885	135 231 7511	154.100 4636	175.950 5446	230.632 2397	304.243 5234
43	125 276 4040	142.993 3387	163.575 9891	187.507 5772	247.776 4695	329.583 0053
44	131.913 8422	151.143 0056	173.572 6685	199.758 0319	266.120 8513	356.949 6457
45	138.849 9651	159.700 1559	184.119 1653	212.743 5138	285.749 3108	386.505 6174
46	146.098 2135	168.685 1637	195.245 7194	226.508 1246	306.751 7626	418.426 0668
47	153.672 6331	178.119 4219	206.984 2339	241.098 6121	329.224 3860	452.900 1521
48	161.587 9016	188 025 3929	219.368 3668	256.564 5288	353.270 0930	490.132 1643
49	169.859 3572	198.426 6626	232.433 6270	272.958 4006	378.998 9995	530.342 7374
50	178 503 0283	209 347 9957	246.217 4765	290.335 9046	406.528 9295	573.770 1564

Table 5

PRESENT VALUE OF ANNUITY OF 1

$$a_{\overline{n}|i} = \frac{1 - \dfrac{1}{(1+i)^n}}{i}$$

n	½%	1%	1¼%	1½%	1¾%	2%
1	0.995 0249	0.990 0990	0.987 6543	0.985 2217	0.982 8010	0.980 3922
2	1.985 0994	1.970 3951	1.963 1154	1.955 8834	1.948 6988	1.941 5609
3	2.970 2481	2.940 9852	2.926 5337	2.912 2004	2.897 9840	2.883 8833
4	3.950 4957	3.901 9656	3.878 0580	3.854 3847	3.830 9425	3.807 7287
5	4.925 8663	4.853 4312	4.817 8350	4.782 6450	4.747 8551	4.713 4595
6	5.896 3844	5.795 4765	5.746 0099	5.697 1872	5.648 9976	5.601 4309
7	6.862 0740	6.728 1945	6.662 7258	6.598 2140	6.534 6411	6.471 9911
8	7.822 9592	7.651 6778	7.568 1243	7.485 9251	7.405 0530	7.325 4814
9	8.779 0639	8.566 0176	8.462 3450	8.360 5173	8.260 4943	8.162 2367
10	9.730 4119	9.471 3045	9.345 5259	9.222 1846	9.101 2229	8.982 5850
11	10.677 0267	10.367 6282	10.217 8034	10.071 1178	9.927 4918	9.786 8481
12	11.618 9321	11.255 0775	11.079 3120	10.907 5052	10.739 5497	10.575 3412
13	12.556 1513	12.133 7401	11.930 1847	11.731 5322	11.537 6410	11.348 3738
14	13.488 7078	13.003 7030	12.770 5528	12.543 3815	12.322 0059	12.106 2488
15	14.416 6246	13.865 0525	13.600 5459	13.343 2330	13.092 8805	12.849 2635
16	15.339 9250	14.717 8738	14.420 2923	14.131 2641	13.850 4968	13.577 7093
17	16.258 6319	15.562 2513	15.229 9183	14.907 6493	14.595 0828	14.291 8719
18	17.172 7680	16.398 2686	16.029 5489	15.672 5609	15.326 8627	14.992 0313
19	18.082 3562	17.226 0085	16.819 3076	16.426 1684	16.046 0567	15.678 4620
20	18.987 4192	18.045 5530	17.599 3161	17.168 6388	16.752 8813	16.351 4333
21	19.887 9792	18.856 9831	18.369 6950	17.900 1367	17.447 5492	17.011 2092
22	20.784 0590	19.660 3793	19.130 5629	18.620 8244	18.130 2695	17.658 0482
23	21.675 6806	20.455 8211	19.882 0374	19.330 8615	18.801 2476	18.292 2041
24	22.562 8662	21.243 3873	20.624 2345	20.030 4054	19.460 6857	18.913 9256
25	23.445 6380	22.023 1557	21.357 2686	20.719 6112	20.108 7820	19.523 4565
26	24.324 0179	22.795 2037	22.081 2530	21.398 6317	20.745 7317	20.121 0358
27	25.198 0278	23.559 6076	22.796 2992	22.067 6175	21.371 7264	20.706 8978
28	26.067 6894	24.316 4132	23.502 5178	22.726 7167	21.986 9547	21.281 2721
29	26.933 0242	25.065 7853	24.200 0176	23.376 0756	22.591 6017	21.844 3847
30	27.794 0540	25.807 7082	24.888 9062	24.015 8380	23.185 8493	22.396 4556
31	28.650 8000	26.542 2854	25.569 2901	24.646 1458	23.769 8765	22.937 7015
32	29.503 2836	27.269 5895	26.241 2742	25.267 1387	24.343 8590	23.468 3348
33	30.351 5259	27.989 6926	26.904 9622	25.878 9544	24.907 9695	23.988 5636
34	31.195 5482	28.702 6659	27.560 4564	26.481 7285	25.462 3779	24.498 5917
35	32.035 3713	29.408 5801	28.207 8582	27.075 5946	26.007 2510	24.998 6193
36	32.871 0162	30.107 5050	28.847 2674	27.660 6843	26.542 7528	25.488 8425
37	33.702 5037	30.799 5099	29.478 7826	28.237 1274	27.069 0446	25.969 4534
38	34.529 8544	31.484 6633	30.102 5013	28.805 0516	27.586 2846	26.440 6406
39	35.353 0890	32.163 0330	30.718 5198	29.364 5829	28.094 6286	26.902 5888
40	36.172 2279	32.834 6861	31.326 9332	29.915 8452	28.594 2296	27.355 4792
41	36.987 2914	33.499 6892	31.927 8352	30.458 9608	29.085 2379	27.799 4895
42	37.798 2999	34.158 1081	32.521 3187	30.994 0500	29.567 8014	28.234 7936
43	38.605 2735	34.810 0081	33.107 4753	31.521 2316	30.042 0652	28.661 5623
44	39.408 2324	35.455 4535	33.686 3954	32.040 6222	30.508 1722	29.079 9631
45	40.207 1964	36.094 5084	34.258 1682	32.552 3372	30.966 2626	29.490 1599
46	41.002 1855	36.727 2361	34.822 8822	33.056 4898	31.416 4743	29.892 3136
47	41.793 2194	37.353 6991	35.380 6244	33.553 1920	31.858 9428	30.286 5820
48	42.580 3178	37.973 9595	35.931 4809	34.042 5537	32.293 8013	30.673 1196
49	43.363 5003	38.588 0787	36.475 5367	34.524 6834	32.721 1806	31.052 0780
50	44.142 7864	39.196 1175	37.012 8757	34.999 6881	33.141 2095	31.423 6059

$$a_{\overline{n}|i} = \frac{1 - \dfrac{1}{(1+i)^n}}{i}$$

n	2¼%	2½%	2¾%	3%	3½%	4%
1	0.977 9951	0.975 6098	0.973 2360	0.970 8738	0 966 1836	0.961 5385
2	1.934 4696	1.927 4242	1.920 4243	1.913 4697	1.899 6943	1 886 0947
3	2 869 8969	2.856 0236	2.842 2621	2 828 6114	2 801 6370	2 775 0910
4	3.784 7402	3.761 9742	3.739 4279	3.717 0984	3.673 0792	3.629 8952
5	4.679 4525	4.645 8285	4.612 5819	4.579 7072	4.515 0524	4.451 8223
6	5.554 4768	5.508 1254	5.462 3668	5.417 1914	5 328 5530	5 242 1369
7	6.410 2463	6.349 3906	6 289 4081	6 230 2830	6 114 5440	6 002 0547
8	7.247 1846	7.170 1372	7.094 3144	7.019 6922	6.873 9555	6.732 7449
9	8.065 7062	7.970 8655	7.877 6783	7.786 1089	7.607 6865	7.435 3316
10	8.866 2164	8.752 0639	8.640 0762	8.530 2028	8.316 6053	8.110 8958
11	9.649 1113	9.514 2087	9 382 0693	9 252 6241	9 001 5510	8 760 4767
12	10.414 7788	10.257 7646	10.104 2037	9.954 0040	9.663 3343	9.385 0738
13	11 163 5979	10.983 1850	10 807 0109	10.634 9553	10.302 7385	9.985 6479
14	11.895 9392	11.690 0122	11.491 0081	11.296 0731	10.920 5203	10.563 1229
15	12 612 1655	12.381 3777	12.156 6989	11 937 9351	11.517 4109	11.118 3874
16	13.312 6313	13.055 0027	12 804 5732	12 561 1020	12.094 1168	11.652 2956
17	13 997 6834	13.712 1977	13.435 1077	13.166 1185	12.651 3206	12.165 6689
18	14 667 6611	14 353 3636	14.048 7666	13.753 5131	13.189 6817	12.659 2970
19	15.322 8959	14.978 8013	14 646 0016	14.323 7991	13.709 8374	13.133 9394
20	15.963 7124	15.589 1623	15.227 2521	14.877 4749	14.212 4033	13.590 3263
21	16.590 4278	16.184 5486	15.792 9461	15 415 0241	14 697 9742	14.029 1600
22	17.203 3523	16 765 4132	16 343 4999	15.936 9166	15 167 1248	14.451 1153
23	17.802 7896	17 332 1105	16 879 3186	16 443 6084	15 620 4105	14.856 8417
24	18 389 0362	17.884 9858	17.400 7967	16 935 5421	16.058 3676	15.246 9631
25	18.962 3826	18.424 3764	17.908 3180	17 413 1477	16.481 5146	15.622 0799
26	19.523 1126	18.950 6111	18 402 2559	17 876 8424	16.890 3523	15 982 7692
27	20.071 5038	19.464 0109	18.882 9741	18 327 0315	17 285 3645	16 329 5858
28	20.607 8276	19 964 8887	19.350 8264	18 764 1082	17 667 0189	16.663 0632
29	21.132 3498	20.453 5499	19 806 1571	19 188 4546	18 035 7670	16 983 7146
30	21.645 3299	20.930 2926	20.249 3013	19.600 4414	18.392 0454	17.292 0333
31	22.147 0219	21.395 4074	20.680 5852	20 000 4285	18 736 2758	17.588 4936
32	22.637 6742	21.849 1780	21.100 3262	20.388 7655	19.068 8655	17 873 5515
33	23 117 5298	22 291 8809	21.508 8333	20 765 7918	19.390 2082	18 147 6457
34	23.586 8262	22.723 7863	21.906 4071	21.131 8367	19.700 6842	18.411 1978
35	24.045 7958	23.145 1573	22.293 3403	21.487 2201	20.000 6611	18.664 6132
36	24.494 6658	23.556 2511	22.669 9175	21.832 2525	20.290 4938	18.908 2820
37	24.933 6585	23 957 3181	23 036 4161	22 167 2354	20.570 5254	19 142 5788
38	25 362 9912	24.348 6030	23.393 1057	22.492 4616	20.841 0874	19.367 8642
39	25.782 8765	24.730 3444	23 740 2488	22.808 2151	21.102 4999	19.584 4848
40	26.193 5222	25.102 7751	24.078 1011	23.114 7720	21.355 0723	19.792 7739
41	26.595 1317	25.466 1220	24 406 9110	23.412 4000	21.599 1037	19.993 0518
42	26.987 9039	25.820 6068	24.726 9207	23.701 3592	21.834 8828	20.185 6267
43	27.372 0332	26.166 4457	25 038 3656	23 981 9021	22.062 6887	20 370 7949
44	27.747 7097	26 503 8495	25.341 4751	24.254 2739	22.282 7910	20.548 8413
45	28.115 1195	26.833 0239	25.636 4721	24.518 7125	22.495 4503	20.720 0397
46	28.474 4445	27.154 1696	25 923 5738	24.775 4491	22.700 9181	20.884 6536
47	28.825 8626	27.467 4826	26 202 9915	25.024 7078	22.899 4378	21.042 9361
48	29.169 5478	27.773 1537	26.474 9309	25 266 7066	23.091 2443	21.195 1309
49	29.505 6702	28.071 3695	26.739 5922	25 501 6569	23.276 5645	21.341 4720
50	29.834 3963	28.362 3117	26.997 1700	25.729 7640	23.455 6179	21.482 1846

$$a_{\overline{n}|i} = \frac{1 - \dfrac{1}{(1+i)^n}}{i}$$

n	4½%	5%	5½%	6%	7%	8%
1	0.956 9378	0.952 3810	0.947 8673	0.943 3962	0.934 5794	0.925 9259
2	1.872 6678	1.859 4104	1.846 3197	1.833 3927	1.808 0182	1.783 2648
3	2.748 9644	2.723 2480	2.697 9334	2.673 0120	2.624 3160	2.577 0970
4	3.587 5257	3.545 9505	3.505 1501	3.465 1056	3.387 2113	3.312 1268
5	4.389 9767	4.329 4767	4.270 2845	4.212 3638	4.100 1974	3.992 7100
6	5.157 8725	5.075 6923	4.995 5303	4.917 3243	4.766 5397	4.622 8797
7	5.892 7009	5.786 3731	5.682 9671	5.582 3814	5.389 2894	5.206 3701
8	6.595 8861	6.463 2128	6.334 5660	6.209 7938	5.971 2985	5.746 6389
9	7.268 7905	7.107 8217	6.952 1953	6.801 6923	6.515 2323	6.246 8879
10	7.912 7182	7.721 7349	7.537 6258	7.360 0871	7.023 5816	6.710 0814
11	8.528 9169	8.306 4142	8.092 5363	7.886 8746	7.498 6744	7.138 9643
12	9.118 5808	8.863 2516	8.618 5179	8.383 8439	7.942 6863	7.536 0780
13	9.682 8524	9.393 5730	9.117 0785	8.852 6830	8.357 6508	7.903 7759
14	10.222 8253	9.898 6409	9.589 6479	9.294 9839	8.745 4680	8.244 2370
15	10.739 5457	10.379 6580	10.037 5809	9.712 2490	9.107 9140	8.559 4787
16	11.234 0151	10.837 7696	10.462 1620	10.105 8953	9.446 6486	8.851 3692
17	11.707 1914	11.274 0663	10.864 6086	10.477 2597	9.763 2230	9.121 6381
18	12.159 9918	11.689 5869	11.246 0745	10.827 6035	10.059 0869	9.371 8871
19	12.593 2936	12.085 3209	11.607 6535	11.158 1165	10.335 5953	9.603 5992
20	13.007 9365	12.462 2103	11.950 3825	11.469 9212	10.594 0143	9.818 1474
21	13.404 7239	12.821 1527	12.275 2441	11.764 0766	10.835 5273	10.016 8032
22	13.784 4248	13.163 0026	12.583 1697	12.041 5817	11.061 2405	10.200 7437
23	14.147 7749	13.488 5739	12.875 0424	12.303 3790	11.272 1874	10.371 0590
24	14.495 4784	13.798 6418	13.151 6990	12.550 3575	11.469 3340	10.528 7583
25	14.828 2090	14.093 9446	13.413 9327	12.783 3562	11.653 5832	10.674 7762
26	15.146 6115	14.375 1853	13.662 4954	13.003 1662	11.825 7787	10.809 9780
27	15.451 3028	14.643 0336	13.898 0999	13.210 5341	11.986 7091	10.935 1648
28	15.742 8735	14.898 1273	14.121 4217	13.406 1643	12.137 1113	11.051 0785
29	16.021 8885	15.141 0736	14.333 1012	13.590 7210	12.277 6741	11.158 4060
30	16.288 8885	15.372 4510	14.533 7452	13.764 8312	12.409 0412	11.257 7833
31	16.544 3910	15.592 8105	14.723 9291	13.929 0860	12.531 8142	11.349 7994
32	16.788 8909	15.802 6767	14.904 1982	14.084 0434	12.646 5553	11.434 9994
33	17.022 8621	16.002 5492	15.075 0694	14.230 2296	12.753 7900	11.513 8884
34	17.246 7580	16.192 9040	15.237 0326	14.368 1411	12.854 0094	11.586 9337
35	17.461 0124	16.374 1943	15.390 5522	14.498 2464	12.947 6723	11.654 5682
36	17.666 0406	16.546 8517	15.536 0684	14.620 9871	13.035 2078	11.717 1928
37	17.862 2398	16.711 2873	15.673 9985	14.736 7803	13.117 0166	11.775 1785
38	18.049 9902	16.867 8927	15.804 7379	14.846 0192	13.193 4735	11.828 8690
39	18.229 6557	17.017 0407	15.928 6615	14.949 0747	13.264 9285	11.878 5824
40	18.401 5844	17.159 0864	16.046 1247	15.046 2969	13.331 7089	11.924 6133
41	18.566 1095	17.294 3680	16.157 4642	15.138 0159	13.394 1204	11.967 2346
42	18.723 5498	17.423 2076	16.262 9992	15.224 5433	13.452 4490	12.006 6987
43	18.874 2103	17.545 9120	16.363 0324	15.306 1729	13.506 9617	12.043 2395
44	19.018 3831	17.662 7733	16.457 8506	15.383 1820	13.557 9081	12.077 0736
45	19.156 3474	17.774 0698	16.547 7257	15.455 8321	13.605 5216	12.108 4015
46	19.288 3707	17.880 0665	16.632 9154	15.524 3699	13.650 0202	12.137 4088
47	19.414 7088	17.981 0157	16.713 6639	15.589 0282	13.691 6077	12.164 2674
48	19.535 6065	18.077 1578	16.790 2027	15.650 0266	13.730 4744	12.189 1365
49	19.651 2981	18.168 7217	16.862 7514	15.707 5723	13.766 7986	12.212 1634
50	19.762 0078	18.255 9255	16.931 5179	15.761 8606	13.800 7463	12.233 4846

Table 6

RENT OF PRESENT VALUE OF ANNUITY OF 1

$$\frac{1}{a_{\overline{n}|i}} = \frac{1}{\dfrac{1 - \dfrac{1}{(1+i)^n}}{i}}$$

n	½%	1%	1¼%	1½%	1¾%	2%
1	1.005 0000	1.010 0000	1.012 5000	1.015 0000	1.017 5000	1.020 0000
2	0.503 7531	0.507 5124	0.509 3944	0.511 2779	0.513 1630	0.515 0495
3	0.336 6722	0.340 0221	0.341 7012	0.343 3830	0.345 0675	0.346 7547
4	0.253 1328	0.256 2811	0.257 8610	0.259 4448	0.261 0324	0.262 6238
5	0.203 0100	0.206 0398	0.207 5621	0.209 0893	0.210 6214	0.212 1584
6	0.169 5955	0.172 5484	0.174 0338	0.175 5252	0.177 0226	0.178 5258
7	0.145 7285	0.148 6283	0.150 0887	0.151 5562	0.153 0306	0.154 5120
8	0.127 8289	0.130 6903	0.132 1331	0.133 5840	0.135 0429	0.136 5098
9	0.113 9074	0.116 7404	0.118 1706	0.119 6098	0.121 0581	0.122.5154
10	0.102 7706	0.105 5821	0.107 0031	0.108 4342	0.109 8754	0.111 3265
11	0.093 6590	0.096 4541	0.097 8684	0.099 2938	0.100 7304	0.102 1779
12	0.086 0664	0.088 8488	0.090 2583	0.091 6800	0.093 1138	0.094 5596
13	0.079 6422	0.082 4148	0.083 8210	0.085 2404	0.086 6728	0.088 1184
14	0.074 1361	0.076 9012	0.078 3052	0.079 7233	0.081 1556	0.082 6020
15	0.069 3644	0.072 1238	0.073 5265	0.074 9444	0.076 3774	0.077 8255
16	0.065 1894	0.067 9446	0.069 3467	0.070 7651	0.072 1996	0.073 6501
17	0.061 5058	0.064 2581	0.065 6602	0.067 0797	0.068 5162	0.069 9698
18	0.058 2317	0.060 9820	0.062 3848	0.063 8058	0.065 2449	0.066 7021
19	0.055 3025	0.058 0518	0.059 4555	0.060 8785	0.062 3206	0.063 7818
20	0.052 6664	0.055 4153	0.056 8204	0.058 2457	0.059 6912	0.061 1567
21	0.050 2816	0.053 0308	0.054 4375	0.055 8655	0.057 3146	0.058 7848
22	0.048 1138	0.050 8637	0.052 2724	0.053 7033	0.055 1564	0.056 6314
23	0.046 1346	0.048 8858	0.050 2967	0.051 7308	0.053 1880	0.054 6681
24	0.044 3206	0.047 0735	0.048 4866	0.049 9241	0.051 3857	0.052 8711
25	0.042 6519	0.045 4068	0.046 8225	0.048 2635	0.049 7295	0.051 2204
26	0.041 1116	0.043 8689	0.045 2873	0.046 7320	0.048 2027	0.049 6992
27	0.039 6856	0.042 4455	0.043 8668	0.045 3153	0.046 7908	0.048 2931
28	0.038 3617	0.041 1244	0.042 5186	0.044 0011	0.045 4815	0.046 9897
29	0.037 1291	0.039 8950	0.041 3223	0.042 7788	0.044 2642	0.045 7784
30	0.035 9789	0.038 7481	0.040 1785	0.041 6392	0.043 1298	0.044 6499
31	0.034 9030	0.037 6757	0.039 1094	0.040 5743	0.042 0701	0.043 5963
32	0.033 8945	0.036 6709	0.038 1079	0.039 5771	0.041 0781	0.042 6106
33	0.032 9473	0.035 7274	0.037 1679	0.038 6414	0.040 1478	0.041 6865
34	0.032 0559	0.034 8400	0.036 2839	0.037 7619	0.039 2736	0.040 8187
35	0.031 2155	0.034 0037	0.035 4511	0.036 9336	0.038 4508	0.040 0022
36	0.030 4219	0.033 2143	0.034 6653	0.036 1524	0.037 6751	0.039 2329
37	0.029 6714	0.032 4680	0.033 9227	0.035 4114	0.036 9426	0.038 5068
38	0.028 9604	0.031 7615	0.033 2198	0.034 7161	0.036 2499	0.037 8206
39	0.028 2861	0.031 0916	0.032 5536	0.034 0546	0.035 5940	0.037 1711
40	0.027 6455	0.030 4556	0.031 9214	0.033 4271	0.034 9721	0.036 5558
41	0.027 0363	0.029 8510	0.031 3206	0.032 8311	0.034 3817	0.035 9719
42	0.026 4562	0.029 2756	0.030 7491	0.032 2643	0.033 8206	0.035 4173
43	0.025 9032	0.028 7274	0.030 2047	0.031 7247	0.033 2867	0.034 8899
44	0.025 3754	0.028 2044	0.029 6856	0.031 2104	0.032 7781	0.034 3879
45	0.024 8712	0.027 7050	0.029 1901	0.030 7198	0.032 2932	0.033 9096
46	0.024 3889	0.027 2278	0.028 7168	0.030 2512	0.031 8304	0.033 4534
47	0.023 9273	0.026 7711	0.028 2641	0.029 8034	0.031 3884	0.033 0179
48	0.023 4850	0.026 3338	0.027 8307	0.029 3750	0.030 9657	0.032 6018
49	0.023 0609	0.025 9147	0.027 4156	0.028 9618	0.030 5612	0.032 2040
50	0.022 6538	0.025 5127	0.027 0176	0.028 5717	0.030 1739	0.031 8232

Note: For Rent of Annuity $\dfrac{1}{\dfrac{(1+i)^n - 1}{i}}$ subtract the rate per cent.

Example: Rent of 1 at 1½% for 25 years is .0482635 — .015 = .0332635.

$$\frac{1}{a_{\overline{n}|i}} = \frac{1}{\dfrac{1 - \dfrac{1}{(1+i)^n}}{i}}$$

n	2¼%	2½%	2¾%	3%	3½%	4%
1	1.022 5000	1.025 0000	1.027 5000	1.030 0000	1.035 0000	1.040 0000
2	0.516 9376	0.518 8272	0.520 7183	0.522 6108	0.526 4005	0.530 1961
3	0.348 4446	0.350 1372	0.351 8324	0.353 524	0.356 9342	0.360 3485
4	0.264 2189	0.265 8179	0.267 4206	0.269 0271	0.272 2511	0.275 4901
5	0.213 7002	0.215 2469	0.216 7983	0.218 3546	0.221 4814	0.224 6271
6	0.180 0350	0.181 5500	0.183 0708	0.184 5975	0.187 6682	0.190 7619
7	0.156 0003	0.157 4954	0.158 9975	0.160 5064	0.163 5445	0.166 6096
8	0.137 9846	0.139 4674	0.140 9580	0.142 4564	0.145 4767	0.148 5278
9	0.123 9817	0.125 4569	0.126 9410	0.128 4339	0.131 4460	0.134 4930
10	0.112 7877	0.114 2588	0.115 7397	0.117 2305	0.120 2414	0.123 2909
11	0.103 6365	0.105 1060	0.106 5863	0.108 0775	0.111 0920	0.114 1490
12	0.096 0174	0.097 4871	0.098 9687	0.100 4621	0.103 4840	0.106 5522
13	0.089 5769	0.091 0483	0.092 5325	0.094 0295	0.097 0616	0.100 1437
14	0.084 0623	0.085 5365	0.087 0246	0.088 5263	0.091 5707	0.094 6690
15	0.079 2885	0.080 7665	0.082 2592	0.083 7666	0.086 8251	0.089 9411
16	0.075 1166	0.076 5990	0.078 0971	0.079 6109	0.082 6848	0.085 8200
17	0.071 4404	0.072 9278	0.074 4319	0.075 9525	0.079 0431	0.082 1985
18	0.068 1772	0.069 6701	0.071 1806	0.072 7087	0.075 8168	0.078 9933
19	0.065 2618	0.066 7606	0.068 2780	0.069 8139	0.072 9403	0.076 1386
20	0.062 6421	0.064 1471	0.065 6717	0.067 2157	0.070 3611	0.073 5818
21	0.060 2757	0.061 7873	0.063 3194	0.064 8718	0.068 0366	0.071 2801
22	0.058 1282	0.059 6466	0.061 1864	0.062 7474	0.065 9321	0.069 1988
23	0.056 1710	0.057 6964	0.059 2441	0.060 8139	0.064 0188	0.067 3091
24	0.054 3802	0.055 9128	0.057 4686	0.059 0474	0.062 2728	0.065 5868
25	0.052 7360	0.054 2759	0.055 8400	0.057 4279	0.060 6740	0.064 0120
26	0.051 2213	0.052 7688	0.054 3412	0.055 9385	0.059 2054	0.062 5674
27	0.049 8219	0.051 3769	0.052 9578	0.054 5642	0.057 8524	0.061 2385
28	0.048 5253	0.050 0879	0.051 6774	0.053 2932	0.056 6027	0.060 0130
29	0.047 3208	0.048 8913	0.050 4894	0.052 1147	0.055 4454	0.058 8799
30	0.046 1992	0.047 7776	0.049 3844	0.051 0193	0.054 3713	0.057 8301
31	0.045 1528	0.046 7390	0.048 3545	0.049 9989	0.053 3724	0.056 8554
32	0.044 1742	0.045 7683	0.047 3926	0.049 0466	0.052 4415	0.055 9486
33	0.043 2572	0.044 8594	0.046 4925	0.048 1561	0.051 5724	0.055 1036
34	0.042 3966	0.044 0068	0.045 6488	0.047 3220	0.050 7597	0.054 3148
35	0.041 5873	0.043 2056	0.044 8565	0.046 5393	0.049 9984	0.053 5773
36	0.040 8252	0.042 4516	0.044 1113	0.045 8038	0.049 2842	0.052 8869
37	0.040 1064	0.041 7409	0.043 4095	0.045 1116	0.048 6133	0.052 2396
38	0.039 4275	0.041 0701	0.042 7476	0.044 4593	0.047 9821	0.051 6319
39	0.038 7854	0.040 4362	0.042 1226	0.043 8439	0.047 3878	0.051 0608
40	0.038 1774	0.039 8362	0.041 5315	0.043 2624	0.046 8273	0.050 5235
41	0.037 6009	0.039 2679	0.040 9720	0.042 7124	0.046 2982	0.050 0174
42	0.037 0536	0.038 7288	0.040 4418	0.042 1917	0.045 7983	0.049 5402
43	0.036 5336	0.038 2169	0.039 9387	0.041 6981	0.045 3254	0.049 0899
44	0.036 0390	0.037 7304	0.039 4610	0.041 2299	0.044 8777	0.048 6645
45	0.035 5681	0.037 2675	0.039 0069	0.040 7852	0.044 4534	0.048 2625
46	0.035 1192	0.036 8268	0.038 5749	0.040 3625	0.044 0511	0.047 8821
47	0.034 6911	0.036 4067	0.038 1636	0.039 9605	0.043 6692	0.047 5219
48	0.034 2823	0.036 0060	0.037 7716	0.039 5778	0.043 3065	0.047 1807
49	0.033 8918	0.035 6235	0.037 3977	0.039 2131	0.042 9617	0.046 8571
50	0.033 5184	0.035 2581	0.037 0409	0.038 8655	0.042 6337	0.046 5502

Note: For Rent of Annuity $\dfrac{1}{\dfrac{(1+i)^n - 1}{i}}$ subtract the rate per cent.

Example: Rent of 1 at 2% for 20 years is .0611567 — .02 = .0411567.

$$\frac{1}{a_{\overline{n}|i}} = \frac{1}{\dfrac{1 - \dfrac{1}{(1+i)^n}}{i}}$$

n	4½%	5%	5½%	6%	7%	8%
1	1.045 0000	1.050 0000	1.055 0000	1.060 0000	1.070 0000	1.080 0000
2	0.533 9976	0.537 8049	0.541 6180	0.545 4369	0.553 0918	0.560 7692
3	0.363 7734	0.367 2086	0.370 6541	0.374 1098	0.381 0517	0.388 0335
4	0.278 7437	0.282 0118	0.285 2945	0.288 5915	0.295 2281	0.301 9208
5	0.227 7916	0.230 9748	0.234 1764	0.237 3964	0.243 8907	0.250 4564
6	0.193 8784	0.197 0175	0.200 1790	0.203 3626	0.209 7958	0.216 3154
7	0.169 7015	0.172 8198	0.175 9644	0.179 1350	0.185 5532	0.192 0724
8	0.151 6097	0.154 7218	0.157 8640	0.161 0359	0.167 4678	0.174 0148
9	0.137 5745	0.140 6901	0.143 8395	0.147 0222	0.153 4865	0.160 0797
10	0.126 3788	0.129 5046	0.132 6678	0.135 8680	0.142 3775	0.149 0295
11	0.117 2482	0.120 3889	0.123 5707	0.126 7929	0.133 3569	0.140 0763
12	0.109 6662	0.112 8254	0.116 0292	0.119 2770	0.125 9020	0.132 6950
13	0.103 2754	0.106 4558	0.109 6843	0.112 9601	0.119 6509	0.126 5218
14	0.097 8203	0.101 0240	0.104 2791	0.107 5849	0.114 3149	0.121 2968
15	0.093 1138	0.096 3423	0.099 6256	0.102 9628	0.109 7946	0.116 8295
16	0.089 0154	0.092 2699	0.095 5825	0.098 9521	0.105 8577	0.112 9769
17	0.085 4176	0.088 6999	0.092 0420	0.095 4448	0.102 4252	0.109 6294
18	0.082 2369	0.085 5162	0.088 9199	0.092 3565	0.099 4126	0.106 7021
19	0.079 4073	0.082 7450	0.086 1501	0.089 6209	0.096 7530	0.104 1276
20	0.076 8761	0.080 2426	0.083 6793	0.087 1846	0.094 3929	0.101 8522
21	0.074 6006	0.077 9961	0.081 4648	0.085 0046	0.092 2890	0.099 8322
22	0.072 5457	0.075 9705	0.079 4712	0.083 0456	0.090 4058	0.098 0321
23	0.070 6825	0.074 1368	0.077 6696	0.081 2785	0.088 7139	0.096 4222
24	0.068 9870	0.072 4709	0.076 0358	0.079 6790	0.087 1890	0.094 9780
25	0.067 4390	0.070 9525	0.074 5494	0.078 2267	0.085 8105	0.093 6788
26	0.066 0214	0.069 5643	0.073 1931	0.076 9044	0.084 5610	0.092 5071
27	0.064 7195	0.068 2919	0.071 9523	0.075 6972	0.083 4257	0.091 4481
28	0.063 5208	0.067 1225	0.070 8144	0.074 5926	0.082 3919	0.090 4889
29	0.062 4146	0.066 0455	0.069 7686	0.073 5796	0.081 4487	0.089 6185
30	0.061 3915	0.065 0514	0.068 8054	0.072 6489	0.080 5864	0.088 8274
31	0.060 4135	0.064 1321	0.067 9167	0.071 7922	0.079 7969	0.088 1073
32	0.059 5632	0.063 2804	0.067 0952	0.071 0023	0.079 0729	0.087 4508
33	0.058 7445	0.062 4900	0.066 3347	0.070 2729	0.078 4081	0.086 8516
34	0.057 9819	0.061 7554	0.065 6296	0.069 5984	0.077 7967	0.086 3041
35	0.057 2705	0.061 0717	0.064 9749	0.068 9739	0.077 2310	0.085 8033
36	0.056 6058	0.060 4345	0.064 3664	0.068 3948	0.076 7153	0.085 3447
37	0.055 9840	0.059 8398	0.063 7999	0.067 8574	0.076 2369	0.084 9244
38	0.055 4017	0.059 2842	0.063 2722	0.067 3581	0.075 7951	0.084 5389
39	0.054 8557	0.058 7646	0.062 7799	0.066 8938	0.075 3868	0.084 1851
40	0.044 3432	0.058 2782	0.062 3203	0.066 4615	0.075 0091	0.083 8602
41	0.053 8616	0.057 8223	0.061 8909	0.066 0589	0.074 6590	0.083 5615
42	0.053 4087	0.057 3947	0.061 4893	0.065 6834	0.074 3359	0.083 2868
43	0.052 9824	0.056 9933	0.061 1134	0.065 3331	0.074 0359	0.083 0341
44	0.052 5807	0.056 6163	0.060 7613	0.065 0061	0.073 7577	0.082 8015
45	0.052 2020	0.056 2617	0.060 4313	0.064 7005	0.073 4996	0.082 5873
46	0.051 8447	0.055 9282	0.060 1218	0.064 4149	0.073 2600	0.082 3899
47	0.051 5073	0.055 6142	0.059 8313	0.064 1477	0.073 0374	0.082 2080
48	0.051 1886	0.055 3184	0.059 5585	0.063 8977	0.072 8307	0.082 0403
49	0.050 8872	0.055 0397	0.059 3023	0.063 6636	0.072 6385	0.081 8856
50	0.050 6022	0.054 7767	0.059 0615	0.063 4443	0.072 4599	0.081 7429

Note: For Rent of Annuity $\dfrac{1}{\dfrac{(1+i)^n - 1}{i}}$ subtract the rate per cent.

Example: Rent of 1 for 30 years at 3% is .0510193 — .03 = .0210193.

Table 7

AMERICAN EXPERIENCE TABLE OF MORTALITY

(Based on 100,000 living at age of 10)

Age x	Number living l_x	Number dying d_x	Yearly probability of dying q_x	Yearly probability of living p_x
10	100 000	749	0.007 490	0.992 510
11	99 251	746	0.007 516	0.992 484
12	98 505	743	0.007 543	0.992 457
13	97 762	740	0.007 569	0.992 431
14	97 022	737	0.007 596	0.992 404
15	96 285	735	0.007 634	0.992 366
16	95 550	732	0.007 661	0.992 339
17	94 818	729	0.007 688	0 992 312
18	94 089	727	0.007 727	0.992 273
19	93 362	725	0.007 765	0.992 235
20	92 637	723	0.007 805	0.992 195
21	91 914	722	0 007 855	0.992 145
22	91 192	721	0 007 906	0 992 094
23	90 471	720	0.007 958	0 992 042
24	89 751	719	0.008 011	0.991 989
25	89 032	718	0.008 065	0 991 935
26	88 314	718	0.008 130	0 991 870
27	87 596	718	0 008 197	0 991 803
28	86 878	718	0.008 264	0.991 736
29	86 160	719	0.008 345	0.991 655
30	85 441	720	0.008 427	0.991 573
31	84 721	721	0.008 510	0.991 490
32	84 000	723	0.008 607	0 991 393
33	83 277	726	0.008 718	0 991 282
34	82 551	729	0.008 831	0.991 169
35	81 822	732	0 008 946	0 991 054
36	81 090	737	0.009 089	0 990 911
37	80 353	742	0.009 234	0 990 766
38	79 611	749	0.009 408	0.990 592
39	78 862	756	0.009 586	0.990 414
40	78 106	765	0.009 794	0.990 206
41	77 341	774	0.010 008	0 989 992
42	76 567	785	0 010 252	0.989 748
43	75 782	797	0.010 517	0 989 483
44	74 985	812	0.010 829	0.989 171
45	74 173	828	0 011 163	0 988 837
46	73 345	848	0.011 562	0 988 438
47	72 497	870	0.012 000	0.988 000
48	71 627	896	0.012 509	0.987 491
49	70 731	927	0.013 106	0.986 894
50	69 804	962	0.013 781	0 986 219
51	68 842	1 011	0.014 541	0 985 459
52	67 841	1 044	0.015 389	0.984 611
53	66 797	1 091	0.016 333	0.983 667
54	65 706	1 143	0.017 396	0.982 604

AMERICAN EXPERIENCE TABLE OF MORTALITY

Age x	Number living l_x	Number dying d_x	Yearly probability of dying q_x	Yearly probability of living p_x
55	64 563	1 199	0.018 571	0.981 429
56	63 364	1 260	0.019 885	0.980 115
57	62 104	1 325	0.021 335	0.978 665
58	60 779	1 394	0.022 936	0.977 064
59	59 385	1 468	0.024 720	0.975 280
60	57 917	1 546	0.026 693	0.973 307
61	56 371	1 628	0.028 880	0.971 120
62	54 743	1 713	0.031 292	0.968 708
63	53 030	1 800	0.033 943	0.966 057
64	51 230	1 889	0.036 873	0.963 127
65	49 341	1 980	0.040 129	0.959 871
66	47 361	2 070	0.043 707	0.956 293
67	45 291	2 158	0.047 647	0.952 353
68	43 133	2 243	0.052 002	0.947 998
69	40 890	2 321	0.056 762	0.943 238
70	38 569	2 391	0.061 993	0.938 007
71	36 178	2 448	0.067 665	0.932 335
72	33 730	2 487	0.073 733	0.926 267
73	31 243	2 505	0.080 178	0.919 822
74	28 738	2 501	0.087 028	0.912 972
75	26 237	2 476	0.094 371	0.905 629
76	23 761	2 431	0.102 311	0.897 689
77	21 330	2 369	0.111 064	0.888 936
78	18 961	2 291	0.120 827	0.879 173
79	16 670	2 196	0.131 734	0.868 266
80	14 474	2 091	0.144 466	0.855 534
81	12 383	1 964	0.158 605	0.841 395
82	10 419	1 816	0.174 297	0.825 703
83	8 603	1 648	0.191 561	0.808 439
84	6 955	1 470	0.211 359	0.788 641
85	5 485	1 292	0.235 552	0.764 448
86	4 193	1 114	0.265 681	0.734 319
87	3 079	933	0.303 020	0.696 980
88	2 146	744	0.346 692	0.653 308
89	1 402	555	0.395 863	0.604 137
90	847	385	0.454 545	0.545 455
91	462	246	0.532 466	0.467 534
92	216	137	0.634 259	0.365 741
93	79	58	0.734 177	0.265 823
94	21	18	0.857 143	0.142 857
95	3	3	1.000 000	0.000 000

Table 8

COMMUTATION COLUMNS, 3½ PER CENT

Age x	D_x	N_x	C_x	M_x	$1 + a_x$	A_x
10	70891.9	1575 535	513.02	17612.9	22.2245	0.24845
11	67981.5	1504 643	493 69	17099 9	22 1331	0.25154
12	65189.0	1436 662	475 08	16606.2	22.0384	0.25474
13	62509.4	1371 473	457 16	16131.1	21.9403	0.25806
14	59938.4	1308 963	439.91	15674.0	21.8385	0.26151
15	57471.6	1249 025	423.88	15234.1	21.7329	0 26508
16	55104.2	1191 553	407.87	14810.2	21.6236	0 26877
17	52832 9	1136 449	392.47	14402.3	21.5102	0.27261
18	50653.9	1083 616	378.15	14009.8	21 3926	0 27659
19	48562.8	1032 962	364.36	13631.7	21.2707	0.28071
20	46556.2	984 400	351 07	13267.3	21.1443	0 28497
21	44630.8	937 843	338 73	12916 3	21.0134	0 28940
22	42782.8	893 213	326 82	12577 5	20 8779	0 29399
23	41009.2	850 430	315 33	12250.7	20 7375	0.29873
24	39307.1	809 421	304 24	11935.4	20.5922	0.30365
25	37673.6	770 113	293 55	11631.1	20.4417	0 30873
26	36106.1	732 440	283 62	11337.6	20 2858	0 31401
27	34601.5	696 334	274 03	11054 0	20.1244	0.31947
28	33157.4	661 732	264 76	10779 9	19.9573	0.32512
29	31771.3	628 575	256 16	10515.2	19.7843	0 33097
30	30440 8	596 804	247 85	10259.0	19.6054	0 33702
31	29163 5	566 363	239 797	10011 2	19.4202	0.34328
32	27937.5	537 199	232 331	9771.38	19.2286	0 34976
33	26760.5	509 262	225.406	9539 04	19.0304	0.35646
34	25630.1	482 501	218 683	9313 64	18.8256	0.36339
35	24544 7	456 871	212 157	9094.96	18.6138	0 37055
36	23502.5	432 326	206 383	8882 80	18 3949	0 37795
37	22501 4	408 824	200 757	8676 42	18.1688	0 38560
38	21539.7	386 323	195.798	8475.66	17.9354	0.39349
39	20615.5	364 783	190.945	8279.86	17.6946	0.40163
40	19727.4	344 167	186.684	8088.92	17.4461	0 41003
41	18873.6	324 440	182 493	7902.23	17.1901	0 41869
42	18052.9	305 566	178 828	7719 74	16.9262	0 42762
43	17263.6	287 513	175 421	7540.91	16.6543	0 43681
44	16504.4	270 250	172.680	7365.49	16.3744	0 44628
45	15773.6	253 745	170.127	7192.81	16.0867	0.45600
46	15070.0	237 972	168.345	7022 68	15.7911	0 46600
47	14392.1	222 902	166 872	6854.34	15.4878	0.47626
48	13738.5	208 510	166 047	6687.47	15 1770	0 48677
49	13107.9	194 771	165 983	6521.42	14.8591	0.49752
50	12498.6	181 663	166.424	6355 44	14.5346	0.50849
51	11909.6	169 165	167.316	6189.01	14.2041	0 51967
52	11339.5	157 255	168 601	6021.70	13.8679	0 53104
53	10787.4	145 916	170.234	5853.10	13.5264	0 54258
54	10252.4	135 128	172.317	5682.86	13.1801	0 55430

COMMUTATION COLUMNS, 3½ PER CENT

Age	D_x	N_x	C_x	M_x	$1 + a_x$	A_x
55	9733.40	124876	174 646	5510 54	12 8296	0 56615
56	9229.60	115142	177 325	5335 90	12 4753	0 57813
57	8740.17	105912.8	180.167	5158 57	12 1179	0 59022
58	8264.44	97172.6	183 140	4978.40	11.7579	0 60239
59	7801.83	88908.2	186 340	4795.27	11.3958	0.61463
60	7351.65	81106.4	189 604	4608.93	11.0324	0.62692
61	6913.44	73754.7	192 909	4419.32	10 6683	0.63924
62	6486.75	66841.3	196 117	4226 41	10.3043	0.65155
63	6071 27	60354.5	199 109	4030.30	9.9410	0.66383
64	5666.85	54283.3	201.887	3831.19	9.5791	0.67607
65	5273.33	48616.4	204 457	3629.30	9.2193	0 68824
66	4890 55	43343.1	206 522	3424 84	8.8626	0 70030
67	4518.65	38452.5	208 022	3218 32	8.5097	0.71223
68	4157.82	33933.9	208 903	3010 30	8.1615	0 72401
69	3808.32	29776.1	208 858	2801.40	7.8187	0.73560
70	3470.67	25967.7	207 881	2592 54	7.4820	0.74698
71	3145.43	22497.1	205 639	2384 66	7.1523	0 75813
72	2833 42	19351.6	201 851	2179 02	6.8298	0.76904
73	2535 75	16518.2	196 436	1977.17	6 5141	0 77972
74	2253 57	13982.5	189 491	1780 73	6.2046	0 79018
75	1987 87	11728 9	181 253	1591 24	5 9002	0 80048
76	1739 39	9741.03	171 940	1409 99	5.6002	0 81062
77	1508 63	8001 63	161 889	1238 05	5.3039	0 82064
78	1295 73	6493 00	151 2646	1076 158	5.0111	0.83054
79	1100 647	5197.27	140 0891	924 894	4.7220	0 84032
80	923 338	4096.62	128 8801	784 805	4.4368	0 84997
81	763.234	3173.29	116 9588	655 924	4.1577	0 85940
82	620 465	2410 05	104 4881	538 966	3 8843	0 86865
83	494.995	1789.59	91 6152	434 478	3 6154	0 87774
84	386 641	1294.59	78.9565	342 862	3.3483	0 88677
85	294.610	907.95	67.0490	263.906	3.0819	0 89578
86	217 598	613.34	55 8566	196 857	2.8187	0 90468
87	154.383	395.74	45 1992	141.000	2 5634	0 91332
88	103 963	241.36	34 82425	95 8011	2.3216	0 92149
89	65.6231	137.398	25 09929	60 9768	2.0937	0 92920
90	38.3047	71.775	16 82244	35 8775	1.8738	0 93664
91	20.18692	33.4700	10 385393	19 05509	1.6580	0 94393
92	9.11888	13.2831	5 588150	8 66970	1.4567	0 95074
93	3.22236	4.16420	2 285784	3 08155	1.2923	0.95630
94	0.827611	0.94184	0 685393	0.79576	1.1380	0 96152
95	0 114232	0.114232	0 110369	0 110369	1 0000	0.96618

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